Coordinate formalism on Hilbert manifolds

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The formalism of local coordinates on infinite-dimensional Hilbert manifolds is introduced. Images of charts on the manifolds are allowed to belong to arbitrary Hilbert spaces of functions including spaces of generalized functions. The corresponding local coordinate form of algebra of tensor fields on Hilbert manifolds is constructed. A single tensor equation in the formalism is shown to produce a family of functional equations on different spaces of functions. This allows for a "covariant" approach to the theory of generalized functions and suggests a way of using generalized functions in solving linear and nonlinear problems. Examples in linear algebra, differential equations, differential geometry and variational calculus are used to illustrate the results.

1 Introduction

The notion of a *vector*, or, more generally a *tensor* is one of the highlights of the 19th century science. In fact, it allows us to separate in a compact way the effects related to a particular choice of coordinates on a manifold from the coordinate independent, "absolute" properties of quantities described by tensors.

The idea of such a "relativisation" has entered physics in a significant way as a requirement that the laws of physics must be described by tensor equations. With the advances of gauge theory this idea was further extended to include gauge transformations i.e. transformations of coordinates on a fibre bundle.

Although originally tensor fields were introduced on finite dimensional manifolds, the needs of mathematics and physics have led one to consider infinite-dimensional manifolds and tensor fields on them.

The notion of an infinite-dimensional manifold is a direct generalization of its finite dimensional counterpart. Namely, such a manifold is a topological space with a differentiable structure where the charts take values in a fixed infinite-dimensional Banach space. In particular, the charts of a Hilbert manifold take values in an infinite-dimensional separable Hilbert space H. In this case we also say that the manifold is modelled on (or after) H.

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In the most general scenario one could assume that images of charts of a Hilbert manifold belong to a family of Hilbert spaces rather than to a fixed space. This possibility is often neglected as all infinite-dimensional separable Hilbert models are isomorphic. The situation seems to be quite similar to the case of a finite dimensional manifold. The images of charts of the latter belong to n-dimensional vector spaces each isomorphic to R^n. There is, however, an important difference. In applications of infinite-dimensional manifolds we usually work with their specific functional realizations. Unlike columns of numbers representing points of a finite dimensional manifold, functions can have very different analytic properties. As no particular space is large enough to contain all functions useful in applications it becomes important to develop a formalism which would make use of several functional models at once.

In the paper such a formalism on Hilbert manifolds is introduced. We let images of charts on a Hilbert manifold belong to arbitrary Hilbert spaces of functions including spaces of generalized functions. Isomorphisms of these spaces are then used to define transformations of coordinates. The resulting formalism (which we call *functional coordinate*, or simply *coordinate*, or *string* formalism) seems to be the most appropriate generalization of the local coordinate approach to tensor fields in infinite-dimensional case.

The possibility of changing the model space H turns out to be essential in developing a coordinate independent approach to the theory of linear operators and in merging linear algebra on Hilbert spaces with the theory of generalized functions.

In particular, the possibility of changing Hilbert models for the abstract Hilbert space $\mathbf{S}$ means that we can alter a specific functional form of an operator $\mathbf{A}$ acting on $\mathbf{S}$. As a result, functionally different operators in the formalism may represent the same (i.e. coordinate independent) operator on $\mathbf{S}$.

Likewise, by changing a Hilbert model H we alter the analytic properties of functions representing the points of $\mathbf{S}$. In particular, the generalized functions become incorporated into the formalism on equal footing with, say, the square-integrable functions.

Moreover, the applications of the formalism are not at all restricted to linear problems. Analogously to tensor equations on manifolds, the functional tensor equations seem to provide a valuable tool for solving nonlinear problems on Hilbert spaces.

Despite the fact that the formalism requires an infinite number of dimensions, it seems at the same time to indicate a deep new connection between the finite and the infinite-dimensional manifolds. Namely, the choice of a model for an infinite-dimensional manifold (rather than the choice of a chart with values in a fixed model) turns out to be similar to the choice of coordinates on a finite dimensional manifold.

In the paper we concentrate on illustrating the formalism by means of multiple examples. The more formal mathematical treatment of the spectral theorem discussed in section 5 and some other specific results in the paper can be found in [3].

In [4] the reader can find some applications of the formalism to quantum mechanics.

Here is the plan of the paper. In section 2 the main definitions of the functional coordinate formalism are introduced. In particular, the notions of a string manifold, string basis, coordinate structure, and a functional coordinate transformation are given and the laws of transformation of component functions of tensor fields are deduced.

The recipe of how to construct the "large" Hilbert spaces of generalized functions is obtained in section 3. In particular, a simple example of a Hilbert space containing the δ-function and all of its derivatives is introduced and the square of the norm of the δ-function is discussed.

The notion of a generalized eigenvalue problem for a linear operator is analyzed in section 4. It is shown that such a problem can be formulated in a functional tensor form i.e. in the form independent of any particular choice of functional coordinates.

In section 5 the basis of eigenvectors of an operator on the string space (the string basis of eigenvectors) is introduced and its properties are investigated. We also discuss the functional tensor approach to the theory of linear operators on Hilbert spaces and formulate the spectral theorem.

Part of the material in sections 2-5 has appeared first in the on-line publications by the author and, in the context of quantum mechanics, in [4].

In the following four sections the specific types of functional coordinate transformations are considered. In section 6 we consider coordinate transformations preserving locality of the operators. As a particular solution of the locality equations we obtain the Fourier transform.

In section 7 transformations preserving the derivative operator are analyzed. It is shown in particular that the generalized and the smooth solutions of linear differential equations with constant coefficients are related by a change of functional coordinates.

In section 8 we investigate coordinate transformations preserving the operator of multiplication by a function. It is shown that except for trivial cases such an operator cannot be preserved.

More general coordinate transformations are considered in section 9. The examples here show how one could use the formalism in the case of more general linear and nonlinear differential equations.

In the last two sections we discuss another interesting relationship between the finite and the infinite-dimensional manifolds. In section 10 we show how the standard formalism of finite dimensional differential geometry on a Riemannian manifold is related to the introduced coordinate formalism on Hilbert manifolds. Here singular generalized functions play an essential role.

In section 11 we deduce the equation of geodesics on a finite dimensional submanifold of a Hilbert manifold by means of the coordinate formalism.

Finally, the results and potential applications are summarized in the conclusion.

2 Coordinate formalism on abstract Hilbert space

As already discussed, we want to incorporate the arbitrary infinite-dimensional separable Hilbert spaces of functions into the structure of a Hilbert manifold. Such spaces will be simply called Hilbert spaces throughout the paper. As all separable Hilbert spaces are isomorphic, it is possible to think that there is a single abstract Hilbert space $\mathbf{S}$. Different Hilbert spaces of functions will be then *realizations* of $\mathbf{S}$. We accept the following definitions:

Definition. A *string space* $\mathbf{S}$ is an abstract infinite-dimensional linear topological space isomorphic (that is, topologically linearly isomorphic) to a separable Hilbert space.

Definition. The elements of $\mathbf{S}$ are called strings and will be denoted by the capital Greek letters $\Phi, \Psi, ...$.

Definition. A *Hilbert space of functions* is either a Hilbert space H, elements of which are equivalence classes of maps between two given subsets of R^n or the Hilbert space H^* dual to H. Two maps f, g are called *equivalent* if the norm of $f - g$ in H is zero.

Definition. Any Hilbert space of functions H is called a *coordinate space*. Elements of H are called *coordinates* (or *functional coordinates*) of strings and will be denoted by the small Greek letters $\varphi, \psi, ...$.

To identify strings with their coordinates we need a linear map $e_H : H \longrightarrow \mathbf{S}$ from a Hilbert space of functions H into the string space $\mathbf{S}$. The action of e_H on $\varphi \in H$ will be written in one of the following ways:

$$e_H(\varphi) = \int e_H(k)\varphi(k)dk = e_{Hk}\varphi^k. \tag{2.1}$$

The integral sign here is used as a notation for the action of e_H on elements of H and in general does not refer to an actual integration. We also use an obvious and convenient generalization of the Einstein's summation convention over the repeated indices k one of which is above and one below. Once again, only in special cases does this notation refer to an actual summation or integration over k.

Definition. A linear isomorphism e_H from a Hilbert space H of functions onto $\mathbf{S}$ is called a *string basis* (or a *functional basis*) on $\mathbf{S}$. The inverse map $e_H^{-1} : \mathbf{S} \longrightarrow H$ is called a *linear coordinate system on* $\mathbf{S}$ (or a *linear functional coordinate system*).

By definition any string Φ is the image of a unique element $a \in H$, i.e.

$$\Phi = e_H(a) \tag{2.2}$$

for a unique $a \in H$. Also, if

$$e_H(a) = 0, \tag{2.3}$$

then $a = 0$.

Assume for example that $H = l_2$, where l_2 is the Hilbert space of square-summable sequences. Then the action of e_H on H can be written as the matrix multiplication of a row $(e_1, e_2, ...)$ of linearly independent vectors in $\mathbf{S}$ by the column of components of $\varphi \in l_2$. In this case the string basis can be identified with the basis in the ordinary sense.

It is worth noticing that the basis e_H defines the space H itself. In fact, it acquires the meaning only as a map on H.

Let $\mathbf{S}^*$ be the dual string space. That is, $\mathbf{S}^*$ is the space of all linear continuous functionals on strings.

Definition. A linear isomorphism of H^* onto $\mathbf{S}^*$ is called a *string basis on* $\mathbf{S}^*$.

We will denote such a basis by e_{H^*}. Decomposition of an element $F \in \mathbf{S}^*$ with respect to the basis will be written in one of the following ways:

$$F = e_{H^*}(f) = \int e_{H^*}(k) f(k) dk = e^k_{H^*} f_k. \tag{2.4}$$

Definition. The basis e_{H^*} is *dual* to the basis e_H if for any string $\Phi = e_{Hk}\varphi^k$ and for any functional $F = e^k_{H^*} f_k$ the following is true:

$$F(\Phi) = f(\varphi). \tag{2.5}$$

In general case we have

$$F(\Phi) = e_{H^*} f(e_H \varphi) = e^*_H e_{H^*} f(\varphi), \tag{2.6}$$

where $e^*_H : \mathbf{S}^* \longrightarrow H^*$ is the adjoint of e_H. Therefore, e_{H^*} is the dual string basis if $e^*_H e_{H^*} : H^* \longrightarrow H^*$ is the identity operator. In this case we will also write

$${e_{Hl}}^* e^k_{H^*} = \delta^k_l. \tag{2.7}$$

In special cases δ^k_l is the usual Krœnecker symbol or the δ- function.

Notice also that the action of F on Φ in any bases e_H on $\mathbf{S}$ and e_{H^*} on $\mathbf{S}^*$ can be written in the following way:

$$F(\Phi) = e^k_{H^*} f_k e_{Hl} \varphi^l = G(f, \varphi) = g^k_l f_k \varphi^l, \tag{2.8}$$

where G is a non-degenerate bilinear functional on $H^* \times H$.

By definition the string space $\mathbf{S}$ is isomorphic to a separable Hilbert space. We can assume then that $\mathbf{S}$ itself is an abstract Hilbert space. Any linear isomorphism $\pi_H : \mathbf{S} \longrightarrow H$ of $\mathbf{S}$ into a Hilbert space of functions can be thought of as inducing the Hilbert structure on H. Respectively, we will assume that the string bases $e_H = {\pi_H}^{-1}$ are isomorphisms of Hilbert spaces. That is, a Hilbert structure on any coordinate space H will be assumed to be induced by a choice of string basis.

Let us assume that H is a real Hilbert space (generalization to the case of a complex Hilbert space will be obvious). We have:

$$(\Phi, \Psi)_S = \mathbf{G}(\Phi, \Psi) = G(\varphi, \psi) = g_{kl}\varphi^k\psi^l, \tag{2.9}$$

where $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ is a bilinear form defining the inner product on $\mathbf{S}$ and $G : H \times H \longrightarrow R$ is the induced bilinear form. The expression on the right is a convenient form of writing the action of G on $H \times H$.

It is useful to notice that the choice of a coordinate Hilbert space determines the corresponding string basis up to a unitary transformation.

Definition. A string basis e_H in $\mathbf{S}$ will be called *orthogonal* if for any $\Phi, \Psi \in \mathbf{S}$

$$(\Phi, \Psi)_S = f_\varphi(\psi), \tag{2.10}$$

where $f_\varphi = (\varphi, \cdot)$ is a *regular* functional and $\Phi = e_H\varphi$, $\Psi = e_H\psi$ as before. That is,

$$(\Phi, \Psi)_S = f_\varphi(\psi) = \int \varphi(x)\psi(x)d\mu(x), \tag{2.11}$$

where $\int$ here denotes an actual integral over a μ-measurable set $D \in R^n$ which is the domain of definition of functions in H.

If the integral in (2.11) is the usual Lebesgue integral and/or a sum over a discrete index x, the corresponding coordinate space will be called an L_2-space. In this case we will also say that the basis e_H is *orthonormal.*

If the integral is a more general Lebesgue-Stieltjes integral, the coordinate space defined by (2.11) will be called an L_2-space with the weight μ.

The bilinear form $\mathbf{G} : \mathbf{S} \times \mathbf{S} \longrightarrow R$ generates a linear isomorphism $\widehat{\mathbf{G}} : \mathbf{S} \longrightarrow \mathbf{S}^*$ by $\mathbf{G}(\Phi, \Psi) = (\widehat{\mathbf{G}}\Phi, \Psi)$. In any basis e_H we have

$$(\Phi, \Psi)_S = (\widehat{\mathbf{G}}e_H\varphi, e_H\psi) = e_H^*\widehat{\mathbf{G}}e_H\varphi(\psi) = \widehat{G}\varphi(\psi), \tag{2.12}$$

where $\widehat{G} = e_H^*\widehat{\mathbf{G}}e_H$ maps H onto H^*. If e_H is orthogonal, then $\widehat{G}\varphi = f_\varphi$. In the case of an orthonormal basis e_H we will also write

$$(\Phi, \Psi)_S = \delta_{kl}\varphi^k\psi^l. \tag{2.13}$$

In a special case δ_{kl} can be the Krœnecker symbol or Dirac δ-function $\delta(k - l)$.

The form of writing in (2.13) makes it especially clear why the basis in the definition above is called orthonormal. In fact, in this case the kernel of the metric functional is a generalization of the Krœnecker symbol, which represents the Euclidean metric in an orthonormal basis.

It follows from the definition that if e_H is orthogonal, then H is a space $L_2(D, \mu)$ of square-integrable functions on a μ-measurable set $D \in R^n$. In particular, not every coordinate Hilbert space H can produce an orthogonal string basis e_H. Assume, for example, that H is a space of functions on R which contains the δ-function as a coordinate φ of a string $\Phi \in \mathbf{S}$ (an example of such a space will be given below in section 3). Then, as $\int \delta(k-l)\varphi(k)\varphi(l)dkdl$ is not defined, the basis e_H can not be orthonormal. That is, the δ-function can not be the coordinate function of a string in orthonormal basis.

On the other hand, Hilbert spaces l_2 and $L_2(R)$ are examples of coordinate spaces that admit an orthonormal string basis.

This result does not contradict the well known existence of an orthonormal basis in any separable Hilbert space. In fact, the meaning of a string basis is quite different from the meaning of an ordinary basis on a Hilbert space. Namely, a string basis on $\mathbf{S}$ permits us to represent an invariant with respect to functional transformations object (string) in terms of a function, which is an element of a Hilbert space of functions. A basis on the latter space in turn permits us to represent this function in terms of numbers, components of the function in the basis.

Notice that formula (2.13) suggests that if H possesses an orhonormal basis e_H, then the chart $(\mathbf{S}, e_H^{-1})$ is analogous to a rectangular Cartesian coordinate systems in Euclidean space. More general formula (2.9) suggests that other string bases produce analogues of oblique Cartesian coordinate systems in Euclidean space.

Definition. A linear coordinate transformation on $\mathbf{S}$ is an isomorphism $\omega : \widetilde{H} \longrightarrow H$ of Hilbert spaces which defines a new string basis $e_{\widetilde{H}} : \widetilde{H} \longrightarrow \mathbf{S}$ by $e_{\widetilde{H}} = e_H \circ \omega$.

Let φ be coordinate of a string Φ in the basis e_H and $\widetilde{\varphi}$ its coordinate in the basis $e_{\widetilde{H}}$. Then $\Phi = e_H\varphi = e_{\widetilde{H}}\widetilde{\varphi} = e_H\omega\widetilde{\varphi}$. That is, $\varphi = \omega\widetilde{\varphi}$ by uniqueness of the decomposition. This provides the transformation law of string coordinates.

Let now $\Phi, \Psi \in \mathbf{S}$ and let $\mathbf{A}$ be a linear operator on $\mathbf{S}$. Let $\Phi = e_H\varphi, \Psi = e_H\psi$ with $\varphi, \psi \in H$. The scalar product $(\Phi, \mathbf{A}\Psi)_S$ is independent of a basis and in a basis e_H reduces to

$$(\varphi, A\psi)_H = (\widehat{G}\varphi, A\psi), \tag{2.14}$$

where $\widehat{G} : H \longrightarrow H^*$ defines the metric on H.

If $\omega : \widetilde{H} \longrightarrow H$ is a linear coordinate transformation and $\varphi = \omega\widetilde{\varphi}, \psi = \omega\widetilde{\psi}$, then

$$(\widehat{G}\varphi, A\psi) = (\widehat{G}\omega\widetilde{\varphi}, A\widetilde{\psi}) = (\omega^*\widehat{G}\omega\widetilde{\varphi}, \omega^{-1}A\omega\widetilde{\psi}) = (\widehat{G}_{\widetilde{H}}\widetilde{\varphi}, A_{\widetilde{H}}\widetilde{\psi}). \tag{2.15}$$

Therefore we have the following transformation laws:

$$\varphi = \omega\widetilde{\varphi} \tag{2.16}$$
$$\psi = \omega\widetilde{\psi} \tag{2.17}$$
$$\widehat{G}_{\widetilde{H}} = \omega^*\widehat{G}\omega \tag{2.18}$$
$$A_{\widetilde{H}} = \omega^{-1}A\omega. \tag{2.19}$$

More generally, consider an arbitrary Hilbert manifold S modelled on $\mathbf{S}$. Let (U_α, π_α) be an atlas on S.

Definition. A collection of quadruples $(U_\alpha, \pi_\alpha, \omega_\alpha, H_\alpha)$, where each H_α is a Hilbert space of functions and ω_α is an isomorphism of **S** onto H_α is called a *functional atlas* on S. A collection of all compatible functional atlaces on **S** is called a *coordinate structure* on S.

Definition. Let (U_α, π_α) be a chart on S. If $p \in U_\alpha$, then $\omega_\alpha \circ \pi_\alpha(p)$ is called the *coordinate* of p. The map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ is called a *coordinate system.* The isomorphisms $\omega_\beta \circ \pi_\beta \circ (\omega_\alpha \circ \pi_\alpha)^{-1} : \omega_\alpha \circ \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \omega_\beta \circ \pi_\beta(U_\alpha \cap U_\beta)$ are called *functional (or string) coordinate transformations.*

As S is a differentiable manifold one can also introduce the tangent bundle structure $\tau : TS \longrightarrow S$ and the bundle $\tau^r_s : T^r_s S \longrightarrow S$ of tensors of rank (r, s).

A coordinate structure on a Hilbert manifold permits one to obtain a functional description of any tensor. Namely, let $\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s)$ be an (r, s)-tensor on S.

Definition. The coordinate map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ for each $p \in U_\alpha$ yields the linear map of tangent spaces $d\rho_\alpha : T_{\omega_\alpha \circ \pi_\alpha(p)} H_\alpha \longrightarrow T_p S$, where $\rho_\alpha = \pi_\alpha^{-1} \circ \omega_\alpha^{-1}$. This map is called a *local coordinate string basis on S*.

Let $e_{H_\alpha} = e_{H_\alpha}(p)$ be such a basis and $e_{H^*_\alpha} = e_{H^*_\alpha}(p)$ be the corresponding dual basis. Notice that for each p the map e_{H_α} is a string basis as defined earlier. Therefore, the local dual basis is defined for each p as before and is a function of p.

We now have $F_i = e_{H^*_\alpha} f_i$, and $\Phi_j = e_{H_\alpha} \varphi_j$ for any $F_i \in T^*_p S$, $\Phi_j \in T_p S$ and some $f_i \in H^*_\alpha, \varphi_j \in H_\alpha$. Therefore

$$\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s) = G_p(f_1, ..., f_r, \varphi_1, ..., \varphi_s) \qquad (2.20)$$

defining component functions of the (r, s)-tensor $\mathbf{G}_p$ in the local coordinate basis e_{H_α}.

In the following sections our goal is to explain the formalism and its usefulness through a variety of examples.

3 Constructing "large" Hilbert spaces of functions

Our first step is to bring the generalized functions (distributions) into the picture. The most common Hilbert spaces containing some singular generalized functions are Sobolev spaces. However, these spaces are rather "meager". In particular, they do not include many of the tempered distributions standardly used in applications.

The following example shows how one can practically construct large Hilbert spaces containing standard spaces of generalized functions as topological subspaces. See [3] for details as well as for other examples of Hilbert spaces of generalized functions.

Example. Let W be the Schwartz space of infinitely differentiable rapidly decreasing functions on R. That is, functions $\varphi \in W$ satisfy inequalities of the form

$\left|x^k\varphi^{(n)}(x)\right| \leq C_{kn}$ for some constants C_{kn} and any $k, n = 0, 1, 2, \ldots$. Let W^* be the dual space of continuous linear functionals on W. Let us find a Hilbert space which contains W^* as a subset. For this consider a linear transformation ρ from the space $L_2(R)$ into W given by

$$(\rho f)(x) = \int f(y)e^{-(x-y)^2-x^2}dy \tag{3.1}$$

for any $f \in L_2(R)$. One can verify that $\rho(L_2(R) \subset W$ and ρ is injective. Moreover, ρ induces the Hilbert metric on $H = \rho(L_2(R))$ by $(\varphi, \psi)_H = (\rho^{-1}\varphi, \rho^{-1}\psi)_{L_2}$. The space H with this metric is Hilbert and the embedding $H \subset W$ is continuous.

It follows in particular that $W^* \subset H^*$ as a set. In fact, any functional continuous on W will be continuous on H. Moreover, by choosing the strong topology on W^* one can show that the embedding of W^* into H^* is continuous. Notice also that one could choose the weak topology on W^* instead as the weak and strong topologies on W^* are equivalent [1].

The method used to obtain the Hilbert space H^* turns out to be quite general. It can be summarized as follows. First, find a linear injection from a standard Hilbert space (say, $L_2(R)$) into itself. This injection induces a Hilbert structure on the image H. Assume the embedding of H into the original Hilbert space is continuous. Then construct the conjugate Hilbert space H^*. By choosing the injection to a small enough subspace H, one obtains as large H^* as one wishes.

The metric on H^* induced by $\rho^* : H^* \longrightarrow L_2^*$ in the example is given by

$$(f, g)_{H^*} = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}f(x)g(z)dydxdz. \tag{3.2}$$

In particular, for the norm of the δ-function we have:

$$(\delta, \delta)_{H^*} = (\rho\rho^*\delta, \delta) = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}\delta(x)\delta(z)dydxdz = \frac{\sqrt{\pi}}{2}. \tag{3.3}$$

The fact that the norm of the δ-function in H^* is finite related to the fact that the metric $g_{xz} = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}dy$ is a smooth function of x and z and is capable of "compensating" singularities of the product of two δ-functions. Conversely, in case of the coordinate space $L_2(R)$ the metric g_{xz} is equal to $\delta(x-z)$. Therefore the norm of the δ-function is not defined. That is, δ-function can not be the coordinate of a string in this case.

More generally, it is easy to see that by "smoothing" the metric we extend the class of (generalized) functions for which the norm defined by this metric is finite. The same is true if we "improve" the behavior of the metric for large $|x|$. Conversely, by "spoiling the metric" we make the corresponding Hilbert space "poor" in terms of a variety of the elements of the space.

4 Functional-tensor equations: Generalized eigenvalue problem

The usefulness of tensor equations in a finite dimensional setting is well recognized. The developed coordinate formalism permits one to introduce the analogous concept of functional tensor equations. Of course, tensor equations on infinite-dimensional manifolds are well known. However, they are ordinary understood in a simplified way as equations independent of a particular choice of a chart taking values in a fixed Hilbert space. When the manifold is a Hilbert space itself, tensor property of equations means simply that equations can be written in a basis independent way. However, as the following example demonstrates, the notion of a string basis generalizes this understanding in a nontrivial way.

Example. Let A be a linear operator on a linear topological space V. If V is a finite-dimensional unitary space and A is, say, Hermitian, then a basis in V exists, such that each vector of it is an eigenvector of A. If V is infinite-dimensional, this statement is no longer true. Yet quite often there exists a complete system of "generalized eigenfunctions" of A in the sense of the definition below (see [2]):

Definition. A linear functional F on V, such that

$$F(A\Phi) = \lambda F(\Phi) \tag{4.1}$$

for every $\Phi \in V$, is called a *generalized eigenfunction of A corresponding to the eigenvalue λ*.

It is usually assumed that V in (4.1) is a space of test functions, i.e. a linear topological space of infinitely differentiable (and "good behaving" at infinity) functions used in the theory of generalized functions.

Notice however, that the definition above makes sense in a more general setting when F and Φ are elements of a pair of dual spaces. Assume then that V is the string space $\mathbf{S}$ and e_H is a string basis on $\mathbf{S}$. Assume F is a generalized eigenfunction of a linear operator $\mathbf{A}$ on $\mathbf{S}$. Then

$$F(\mathbf{A}\Phi) = \lambda F(\Phi), \tag{4.2}$$

that is,

$$e_H^* F(e_H^{-1}\mathbf{A}e_H\varphi) = \lambda e_H^* F(\varphi), \tag{4.3}$$

where $e_H\varphi = \Phi$ and $e_H^{-1}\mathbf{A}e_H$ is the representation of $\mathbf{A}$ in the basis e_H.

By defining $e_H^* F = f$ and $A = e_H^{-1}\mathbf{A}e_H$ we have

$$f(A\varphi) = \lambda f(\varphi). \tag{4.4}$$

Notice that the last equation describes not just one eigenvalue problem, but a family of such problems, one for each string basis e_H. As we change e_H, the operator A in general changes as well, as do the eigenfunctions f.

In particular, let $H \subset L_2(R)$ be a Hilbert space of complex-valued functions such that its dual H^* contains the functionals $f(x) = e^{ipx}$, for any $p \in R$ (see [3] for an example). Assume that the action of the operator of differentiation $A = i\frac{d}{dx}$ on H is defined. The generalized eigenvalue problem for A is

$$f(i\frac{d}{dx}\varphi) = pf(\varphi). \tag{4.5}$$

The equation (4.5) must be satisfied for every φ in H. The functionals

$$f(x) = e^{-ipx} \tag{4.6}$$

are the eigenvectors of A. Let us now consider a coordinate change $\rho : H \longrightarrow \widetilde{H}$ given by the Fourier transform. We have:

$$\psi(k) = (\rho\varphi)(k) = \int \varphi(x)e^{ikx}dx. \tag{4.7}$$

The Fourier transform induces a Hilbert structure on H. Relative to this structure ρ is an isomorphism of the Hilbert spaces $\widetilde{H}$ and H. The inverse transform is given by

$$(\omega\psi)(x) = \frac{1}{2\pi}\int \psi(k)e^{-ikx}dk. \tag{4.8}$$

Notice that as the Fourier image of e^{ipx} is the delta function, the space dual to $\widetilde{H}$ contains the functionals $\delta(k-p)$. According to (4.3) the generalized eigenvalue problem in new coordinates is

$$\omega^* f(\rho A\omega\psi) = p\omega^* f(\psi). \tag{4.9}$$

We have:

$$A\omega\psi = i\frac{d}{dx}\frac{1}{2\pi}\int \psi(k)e^{-ikx}dk = \frac{1}{2\pi}\int k\psi(k)e^{-ikx}dk. \tag{4.10}$$

Therefore,

$$(\rho A\omega\psi)(k) = k\psi(k). \tag{4.11}$$

So, the eigenvalue problem in new coordinates is as follows:

$$g(k\psi) = pg(\psi). \tag{4.12}$$

Thus, we have the eigenvalue problem for the operator of multiplication by the variable. The eigenfunctions here are given by

$$g(k) = \delta(k-p). \tag{4.13}$$

Notice that $g = \omega^* f$ is as it should be. Indeed,

$$(\omega^* f)(k) = \frac{1}{2\pi} \int f(x) e^{ikx} dx = \frac{1}{2\pi} \int e^{-ipx} e^{ikx} dx = \delta(k-p). \tag{4.14}$$

As a result, the eigenvalue problems (4.5), and (4.12) can be considered as two coordinate expressions of a single functional tensor eigenvalue problem

$$F(\mathbf{A}\Phi) = \lambda F(\Phi) \tag{4.15}$$

for an operator $\mathbf{A}$ on $\mathbf{S}$. I

5 The spectral theorem

Having introduced generalized eigenvectors of an operator as elements of a Hilbert space it is natural to ask whether we can make a basis out of them. In what follows we assume that $\mathbf{A}$ is a bounded linear operator on $\mathbf{S}$.

Definition 1. A string basis e_H is the *proper basis* of a linear operator $\mathbf{A}$ on $\mathbf{S}$ with eigenvalues (eigenvalue function) $\lambda = \lambda(k)$, if

$$\mathbf{A} e_H(\varphi) = e_H(\lambda\varphi) \tag{5.1}$$

for any $\varphi \in H$.

As any string basis, the proper basis of $\mathbf{A}$ is a linear map from H onto $\mathbf{S}$ and a numeric function λ is defined on the same set as functions $\varphi \in H$.

By rewriting (5.1) as

$$e_H{}^{-1} \mathbf{A} e_H(\varphi) = \lambda\varphi \tag{5.2}$$

we see that the problem of finding a proper basis of $\mathbf{A}$ is equivalent to the problem of finding such a string basis e_H in which the action of $\mathbf{A}$ reduces to multiplication by a function λ. In the particular case of an l_2-basis this yields the classical problem of finding a basis of eigenvectors of a linear operator.

The operator of multiplication by a function does not always map a Hilbert space H into itself. It is then useful to generalize the definition above.

Notice, that for any $F \in \mathbf{S}^*, \Phi \in \mathbf{S}$ and $\mathbf{A} : \mathbf{S} \longrightarrow \mathbf{S}$ we have

$$F(\mathbf{A}\Phi) = F(\mathbf{A} e_H \varphi) = F(e_{\widetilde{H}} e_{\widetilde{H}}^{-1} \mathbf{A} e_H \varphi) = e_{\widetilde{H}}^{*} F(e_{\widetilde{H}}^{-1} \mathbf{A} e_H \varphi) = f(A\varphi), \tag{5.3}$$

where $f = e_{\widetilde{H}}^{*} F \in \widetilde{H}^*$ and $A : H \longrightarrow \widetilde{H}$ is given by $A = e_{\widetilde{H}}^{-1} \mathbf{A} e_H$.

We then have the following

Definition 2. The representation $A = e_{\widetilde{H}}^{-1} \mathbf{A} e_H$ of $\mathbf{A}$ is called *proper* if for any $F \in \mathbf{S}^*$, $\Phi \in \mathbf{S}$ we have

$$F(\mathbf{A}\Phi) = f(A\varphi) = f(\lambda\varphi), \tag{5.4}$$

where $f = e_{\widetilde{H}}^{*} F$, $\varphi = e_H^{-1} \Phi$ and λ is a function such that the operator of multiplication by λ maps H into $\widetilde{H}$.

If $H = \widetilde{H}$ this is equivalent to the definition 1 above.

It is easy to see that when $H = l_2$ the proper basis and the ordinary basis of eigenvectors of an Hermitian operator $\mathbf{A}$ can be identified. In fact, let e_{l_2} be a proper basis of $\mathbf{A}$ existence of which at this point is assumed. Let $\varphi = (\varphi_1, \varphi_2, ...) \in l_2$. The action of e_{l_2} on φ reduces to the matrix multiplication of the matrix $e_{l_2} = (e_1, e_2, ...)$ of linearly independent vectors by the column of components of φ. That is, $e_{l_2}\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$. Notice that convergence of the series is assured by the continuity of e_{l_2}. We have:

$$\mathbf{A}e_{l_2}\varphi = e_{l_2}(\lambda\varphi) = e_{l_2}(\lambda_1\varphi_1, \lambda_2\varphi_2, ...) = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \quad . \tag{5.5}$$

Assume now that the ordinary basis $e_{l_2} = (e_1, e_2, ...)$ of eigenvectors of $\mathbf{A}$ exists. Then $\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$ and

$$\mathbf{A}(e_1\varphi_1 + e_2\varphi_2 + ...) = \lambda_1 e_1\varphi_1 + \lambda_2 e_2\varphi_2 + ... = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \quad . \tag{5.6}$$

Comparison of (5.5) and (5.6) verifies the claim.

Hermicity of $\mathbf{A}$ leads to a severe restriction on possible spaces H on which the proper basis of $\mathbf{A}$ is defined. In fact, if e_H is a proper basis of an Hermitian operator $\mathbf{A}$ on $\mathbf{S}$, then

$$(\Phi, \mathbf{A}\Psi)_S = (e_H\varphi, \mathbf{A}e_H\psi)_S = (\varphi, \lambda\psi)_H = \int g(x,y)\varphi(x)\lambda(y)\psi(y)dxdy. \tag{5.7}$$

In agreement with notations of section 2, the integral symbol is used here for the action of the bilinear metric functional G with the kernel g. Hermicity gives then

$$\int g(x,y)(\lambda(x) - \lambda(y))\varphi(x)\psi(y)dxdy = 0 \tag{5.8}$$

for any $\varphi, \psi \in H$. Therefore, the bilinear functional defined by $g(x,y)(\lambda(x) - \lambda(y))$ is equal to zero. It follows that $g(x,y) = a(x)\delta(x-y)$ (see [3]).

As we have just seen, when $H = l_2$, the proper basis of an operator $\mathbf{A}$ on $\mathbf{S}$ can be identified with the ordinary basis of eigenvectors of $\mathbf{A}$. The obtained result is then equivalent to the well known orthogonality of the eigenvectors corresponding to different eigenvalues of the Hermitian operator $\mathbf{A}$.

More generally, the result means that the proper bases of Hermitian operators are orthogonal (as defined in section 2). In particular, the space H on which the proper basis of an Hermitian operator is defined must be an L_2-space.

However, many Hermitian operators do not have eigenvectors in L_2-spaces. In particular, from section 4 we recall that the eigenvectors of $A = \lambda\cdot$ acting on $L_2(R)$ can only be defined as functionals f on a space V of functions, such that

$$f(A\varphi) = \lambda_0 f(\varphi) \tag{5.9}$$

for any $\varphi \in V$ and some $\lambda_0 \in R$.

To include such functionals in the formalism we need to consider Hilbert spaces containing "more" functions than $L_2(R)$. By the above this in general requires consideration of non-Hermitian operators.

Example. We know that $A_H = x$ (operator of multiplication by the variable) has no eigenvectors on $L_2(R)$. Consider then a different Hilbert space H of functions of a real variable with a metric G. We have:

$$(G\varphi, A_H\psi) = \int g(k,m)\varphi(k)m\psi(m)dkdm. \tag{5.10}$$

By (5.8)

$$\int g(k,m)\varphi(k)m\psi(m)dkdm \neq \int g(k,m)k\varphi(k)\psi(m)dkdm, \tag{5.11}$$

that is, A_H is not Hermitian unless H is an L_2-space. On the other hand, if $g(k,m)$ is an ordinary function which defines a regular functional, then obviously

$$\int g(k,m)\varphi(k)m\psi(m)dkdm = \int mg(k,m)\varphi(k)\psi(m)dkdm. \tag{5.12}$$

So by taking

$$f = \int g(k,m)\varphi(k)dk, \tag{5.13}$$

we have:

$$f(m\psi) = (mf)(\psi). \tag{5.14}$$

If the functional $f = \int g(k,m)\varphi(k)dk$ above is not regular, we can define the action of A_H on it by taking

$$(mf)(\varphi) = f(m\varphi). \tag{5.15}$$

The resulting operator of multiplication by m is defined on H^* and is self-adjoint in the sense that (5.15) is satisfied for all possible φ, f.

To be specific, assume that H is the Hilbert subspace $H \subset L_2(R)$ of functions as in the example of section 3. The space H consists of infinitely differentiable functions only. The space H^* contains singular distributions. If $\widehat{G}$ is the metric on H, then the inner product on H^* is defined by

$$(f,g)_{H^*} = (\widehat{G}^{-1}f, g) = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}f(x)g(z)dydxdz. \tag{5.16}$$

Unlike the case of $L_2(R)$-space it is possible to find such a function $\varphi \in H$ that $f = \widehat{G}\varphi \in H^*$ in the expression

$$(f, x\psi) \tag{5.17}$$

is a (generalized) eigenvector of x. We can take, for example, $f(x) = \delta(x - \lambda_0)$. With such a choice we have:

$$(f, x\psi) = \lambda_0(f,\psi). \tag{5.18}$$

Let us recall the following definitions:

Definition. Let A be a continuous linear operator which maps a space H into a space $\widetilde{H}$. Then the *adjoint* A^* of operator A maps the space $\widetilde{H}^*$ into the space H^* according to

$$(A^* f, \varphi) = (f, A\varphi) \tag{5.19}$$

for any $\varphi \in H, f \in \widetilde{H}^*$.

Similarly,

Definition. Let A be a continuous linear operator on a Hilbert space H. Then the *Hermitian conjugate* operator A^+ of A is defined on H by

$$(A^+ \varphi, \psi)_H = (\varphi, A\psi)_H, \tag{5.20}$$

for any $\varphi, \psi \in H$.

Let now A be given on H and let $\widehat{G} : H \longrightarrow H^*$ define a metric on H. Then

$$(\widehat{G}\varphi, A\psi) = (A^* \widehat{G}\varphi, \psi) = (\widehat{G}A^+ \varphi, \psi) \tag{5.21}$$

and the relationship between the operators is as follows:

$$A^+ = \widehat{G}^{-1} A^* \widehat{G}. \tag{5.22}$$

Notice that the L_2-spaces are special in that the conjugate of any such space consists of the regular functionals defined by functions $f \in L_2$. It is therefore natural to identify in this case each functional with the function that defines it. Independently, this could be done on any Hilbert space as well. However, such an identification is not preserved under a general transformation of coordinates. In fact, consider a coordinate space H which is a topological subspace of an L_2-space. We then have $H \subset L_2 \subset H^*$. If we agree to identify the common elements of the spaces in the triple, then the identification $L_2 \longrightarrow L_2^*$ by $\varphi \longrightarrow (\varphi, \cdot)_{L_2}$ is inconsistent with the identification $\varphi \longrightarrow (\varphi, \cdot)_H$. The fact that H is a coordinate space shows then that the identification is not preserved under transformations of coordinates. We therefore agree to identify only the L_2-spaces with their conjugates. As for other coordinate spaces we will avoid such an identification.

If H is identified with its conjugate and $A : H \longrightarrow H$ is an operator on H, then the adjoint operator $A^* : H^* \longrightarrow H^*$ and the Hermitian conjugate $A^+ : H \longrightarrow H$ can be identified as well. As we have seen, the identification of H and H^* is not preserved under general transformations of coordinates. This forces us to distinguish between the adjoint and the Hermitian conjugate operators and, respectively, between the self-adjoint and Hermitian operators. Notice that on the L_2-spaces the distinction disappears.

In light of this the following definition will be accepted:

Definition. If A^* is the adjoint of an operator A on H as defined by (5.19) and under the identification $L_2^* = L_2$ the action of A^* and A on their common domain $H \cap H^*$ coincide, we say that the operator A is *self-adjoint*.

Here L_2 denotes the L_2-space of functions having the same domain as functions in H.

It is straightforward to check that if B is a (bounded) Hermitian operator on a L_2-space and $H \subset L_2$ is a topological embedding than the restriction A of B onto H is a self-adjoint operator on H. In fact, A is bounded on H, $H \cap H^* \subset L_2$ and $Af(\varphi) = f(A\varphi)$ for any $f, \varphi \in H \cap H^*$.

According to the definition the operator of multiplication by the variable in the example above, when defined on the entire H, is self-adjoint. The example shows that at least in some cases there is a possibility to accommodate both the Hermicity of an operator in L_2, respectively, the self-adjointness of its restriction A onto a Hilbert subspace $H \subset L_2$ and the inclusion of generalized eigenvectors of A into the conjugate space H^*.

More generally, we have the following theorem (see [3]):

Theorem. Let B be an Hermitian operator on a Hilbert space L_2. Then there exists a topological Hilbert subspace H of L_2 which is dense in L_2 and such that the restriction A of B onto H is a self-adjoint operator and the conjugate space H^* contains the complete set of eigenvectors of A. Moreover, there exists a coordinate transformation $\widetilde{\rho} : H \longrightarrow \widetilde{H}$ such that the transformed operator $\widetilde{A} = \rho A \rho^{-1}$ is the operator of multiplication by x.

In light of this theorem we accept the following definition:

Definition. If a proper basis e_H is such that the complete set of eigenvectors of the operator A in this basis belongs to H^* (alternatively, the complete set of eigenvectors of $\mathbf{A}$ belongs to $\mathbf{S}^*$), then the basis e_H is called the *string basis of eigenvectors of* $\mathbf{A}$ or the *string eigenbasis of* $\mathbf{A}$.

The theorem then asserts that for any Hermitian operator B on a space L_2 one can find a topological dense subspace H of L_2 such that the string eigenbasis e_H of the restriction A of B onto H exists. In particular, for *any* functional $F \in \mathbf{S}^*$ and any string Φ we have $F(\mathbf{A}\Phi) = f(\lambda\varphi_\lambda)$, where φ_λ and f are coordinates of Φ and F in the string eigenbasis e_H and its dual respectively.

The coordinate formalism therefore suggests that for a given functional form of a self-adjoint operator it is important to find an appropriate Hilbert space of functions on which the operator acts. By choosing the space appropriately, one can solve the generalized eigenvalue problem for the operator. Moreover, in this case the corresponding invariant operator $\mathbf{A}$ has a complete set of eigenvectors on $\mathbf{S}$. Respectively, the generalized eigenvalue problem for $\mathbf{A}$ in *any* basis $e_{\widetilde{H}}$ yields a complete set of (generalized) eigenvectors. Further results on the spectral theorem in light of the formalism can be found in [3].

6 Coordinate transformations preserving locality of operators

In section 4 it was shown that a single eigenvalue problem for an operator on the string space leads to a family of eigenvalue problems in particular string bases. Obviously, it is a general feature of tensor equations in the formalism: a specific functional form of an equation depends on the choice of functional coordinates. In the following four sections we will consider various special transformations of functional coordinates and analyze the effect of these transformations on functional operators and equations.

When transforming a functional equation it is often necessary to preserve some of the properties of the equation. In particular, it is often useful to preserve locality of operators in the equation.

Definition. Let H be a coordinate space of functions on R^n. We shall say that a linear operator $A : H \longrightarrow H$ is local if

$$(Af)(x) = \int \sum_{|q|\leq r} a_q(x) D^q \delta(y-x) f(y) dy. \tag{6.1}$$

Here $x, y \in R^n$, r is a nonnegative integer, $q = (q_1, ..., q_n)$ is a set of nonnegative integers, $|q| = q_1 + ... + q_n$, $D^q = \frac{\partial^{|q|}}{\partial {y_1}^{q_1} ... \partial {y_n}^{q_n}}$ and $\delta(y-x)$ denotes the δ-function of the diagonal $x = y$ in R^{2n}.

Assume first that f is an infinitely differentiable function of bounded support and $a_q(x) D^q f(x)$ is integrable. Then formula (6.1) is understood by requiring the validity of "integration by parts" which reduces (6.1) to

$$(Af)(x) = \sum_{|q|\leq r} (-1)^{|q|} a_q(x) D^q f(x). \tag{6.2}$$

More generally, let $f \in H$ be any generalized function on R^n. Then (6.1) is understood by requiring that

$$(Af, \varphi) = (f, B\varphi), \tag{6.3}$$

where φ is any smooth function of bounded support on R^n and

$$(B\varphi)(x) = \sum_{|q|\leq r} D^q (a_q(x) \varphi(x)). \tag{6.4}$$

Here we assume that a_q are smooth functions on R^n.

It is easy to see that locality of an operator is not an invariant property, that is, it depends on a particular choice of coordinates. We now want to describe such coordinate transformations that preserve locality of linear operators.

Suppose then that $A : H \longrightarrow H$ is a local linear operator, $\omega : \widetilde{H} \longrightarrow H$ is a transformation of coordinates, and $A_{\widetilde{H}} = \omega^{-1}A\omega : \widetilde{H} \longrightarrow \widetilde{H}$ is the transformed operator.

The operator $A_{\widetilde{H}}$ will be local if

$$\sum_{|q|\leq r} \omega^{-1}(x,y)a_q(y)D^q\delta(z-y)\omega(z,u)f(u)dydzdu = \sum_{|q|\leq s} b_q(x)D^q\delta(y-x)f(y)dy, \tag{6.5}$$

where notations are as above and the integral symbol is omitted. That is,

$$\sum_{|q|\leq r} a_q(x)D^q\delta(z-x)\omega(z,y)dz = \sum_{|q|\leq s} \omega(x,z)b_q(z)D^q\delta(y-z)dz. \tag{6.6}$$

In the simplest case when H and $\widetilde{H}$ are spaces of functions of one variable and the kernels of A and $A_{\widetilde{H}}$ contain only one term of the form $a_q(x)D^q\delta(y-x)$ each, the equation (6.6) reduces to

$$a(x)\frac{\partial^n}{\partial z^n}\delta(z-x)\omega(z,y)dz = \omega(x,z)b(z)\frac{\partial^m}{\partial y^m}\delta(y-z)dz. \tag{6.7}$$

If particular, when $n=1$ and $m=0$ (6.7) yields

$$a(x)\frac{\partial}{\partial z}\delta(z-x)\omega(z,y)dz = \omega(x,z)b(z)\delta(y-z)dz. \tag{6.8}$$

Assuming that ω is a smooth solution, "integration by parts" gives

$$-a(x)\frac{\partial\omega(x,y)}{\partial x} = \omega(x,y)b(y). \tag{6.9}$$

Solving (6.9) we obtain

$$\omega(x,y) = F(y)e^{-c(x)b(y)}, \tag{6.10}$$

where $c(x) = \int \frac{dx}{a(x)}$ and $F(y)$ is an arbitrary smooth function. To be a coordinate transformation ω must be an isomorphism as well. In particular, the Fourier transform is a solution of (6.9) with

$$\omega(x,y) = e^{ixy}. \tag{6.11}$$

Coordinate transformations satisfying (6.9) preserve locality of the first order differential operators on H by transforming them into operators of multiplication.

In the case of a more general equation (6.7), we have:

$$a(x)\frac{\partial^n\omega(x,y)}{\partial x^n} = (-1)^n\frac{\partial^m(\omega(x,y)b(y))}{\partial y^m}. \tag{6.12}$$

Solutions of (6.12) for different values of n and m produce coordinate transformations preserving locality of various differential operators.

7 Coordinate transformations preserving derivatives

Among solutions of (6.12) those preserving the order q of derivatives are of particular interest. To describe such transformations it is sufficient to obtain solutions of (6.12) with $n = m = 1$. Let us assume here that the coefficients a and b in (6.12) are constants equal to one. Then (6.12) gives the following equation:

$$\frac{\partial \omega(x,y)}{\partial x} + \frac{\partial \omega(x,y)}{\partial y} = 0. \tag{7.1}$$

The smooth solutions of (7.1) are given by

$$\omega(x,y) = f(x-y), \tag{7.2}$$

where f is an arbitrary infinitely differentiable function on R. In particular, the function

$$\omega(x,y) = e^{-(x-y)^2} \tag{7.3}$$

satisfies (7.1). Also, the corresponding transformation is injective (see [3]). When Hilbert structure on $\widetilde{H} = \omega^{-1}(H)$ is induced by ω, this transformation becomes an isomorphism of Hilbert spaces. Therefore, it provides an example of a coordinate transformation that preserves derivatives.

Let now H be a Hilbert space of functions on R^n. We are looking for a nontrivial transformation preserving all partial derivative operators on H. Applying equation (6.6) to this case we obtain assuming $a_q = b_q = 1$:

$$\sum_{i=1}^{n} \frac{\partial \omega(x,y)}{\partial x_i} + \frac{\partial \omega(x,y)}{\partial y_i} = 0. \tag{7.4}$$

The function

$$\omega(x,y) = e^{-(x-y)^2} \tag{7.5}$$

with $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ satisfies (7.4) and the corresponding transformation is injective inducing a Hilbert structure on the space $\widetilde{H} = \omega^{-1}(H)$.

We now have the following

Theorem. The generalized solutions of any linear differential equation with constant coefficients (either ordinary or partial) are coordinate transformations of the corresponding regular solutions (i.e. solutions given by the ordinary functions). That is, let L be a polynomial function of n variables. Let $u, v \in \widetilde{H}$ be functionals on the space K of functions of n variables which are infinitely differentiable and have bounded supports. Assume that u is a generalized solution of

$$L\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right) u = v. \tag{7.6}$$

Then there exists a regular solution φ of

$$L\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)\varphi=\psi, \tag{7.7}$$

where $\varphi=\omega u$, $\psi=\omega v$ and ω is as in (7.5).

To sketch the proof consider first the case of the ordinary differential equation

$$\frac{d}{dx}u(x)=v(x). \tag{7.8}$$

Assume u is a generalized solution of (7.8). Define $\varphi=\omega u$ and $\psi=\omega v$, where ω is as in (7.3). Notice that φ,ψ are infinitely differentiable. In fact, any functional on the space K of infinitely differentiable functions of bounded support acts as follows (see [1]):

$$(f,\varphi)=\int F(x)\varphi^{(m)}(x)dx, \tag{7.9}$$

where F is a continuous function on R. Applying ω to f shows that the result is a smooth function.

As $\omega^{-1}\frac{d}{dx}\omega=\frac{d}{dx}$, we have

$$\omega^{-1}\frac{d}{dx}\omega u=v. \tag{7.10}$$

That is,

$$\frac{d}{dx}\varphi(x)=\psi(x) \tag{7.11}$$

proving the theorem in this case. The higher order derivatives can be treated similarly as

$$\omega^{-1}\frac{d^n}{dx^n}\omega=\omega^{-1}\frac{d}{dx}\omega\omega^{-1}\frac{d}{dx}\omega...\omega^{-1}\frac{d}{dx}\omega. \tag{7.12}$$

That is, transformation ω preserves derivatives of any order. Generalization to the case of several variables is straightforward.

8 Coordinate transformations preserving a product of functions

Let us now investigate changes of equations containing products of functions under transformations of string coordinates. Consider the simplest algebraic equation

$$a(x)f(x)=h(x), \tag{8.1}$$

where f is an unknown (generalized) function of a single variable x and h is an element of a Hilbert space H of functions on a set $D\subset R$. To investigate transformation properties of this equation we need to interpret it as a tensor equation on the string space **S**. The right hand side is a function. Therefore this must be a "vector equation" (i.e. both sides must be $(1,0)$-tensors on the string space). If f is to be a function as well, a must be a $(1,1)$-tensor. That is, the "correct" equation is:

$$a(x)\delta(x-y)f(y)dy = h(x). \tag{8.2}$$

To preserve the product-like form of the equation we need such a coordinate transformation $\omega : \widetilde{H} \longrightarrow H$ that

$$\omega^{-1}(u,x)a(x)\delta(x-y)\omega(y,z)dxdy = b(u)\delta(u-z). \tag{8.3}$$

In this case the equation (8.1) in new coordinates is

$$b(x)\varphi(x) = \psi(x), \tag{8.4}$$

where $h = \omega\psi$, $f = \omega\varphi$, and $\varphi, \psi \in \widetilde{H}$.

Equation (8.3) is a particular case of equation (6.12) with $n = m = 0$. It yields

$$a(x)\omega(x,y) = \omega(x,y)b(y). \tag{8.5}$$

Equation (8.5) is clearly satisfied whenever $a(x) = b(x) = C$, where C is a constant. On another hand, whenever $b'(y) \neq 0$ it is easy to deduce that $\omega(x,y) = a_0(x)\delta(y-x)$, where a_0 is a function (see [3]).

One could refer to the operator $a(x)\delta(x-y)$ in (8.2) as the operator of multiplication by $a(x)$. Clearly, it is a local operator. The obtained result then says that locality of this operator can be preserved only in trivial cases when $a(x) = C$ or ω itself is an operator of multiplication by a function.

In particular, the product of nonconstant functions of one and the same variable is not an invariant operation under a general transformation of coordinates.

On the other hand, consider the equation

$$a(x)f(y) = h(x,y), \tag{8.6}$$

where a and f are functions of a single variable and h is a function of two variables. This equation can be viewed as a tensor equation on the string space. The left hand side represents then a tensor product of two "vectors". The right hand side is a $(2,0)$-tensor. Therefore, any coordinate transformation preserves this form of the equation. In particular, generalized solutions f of this equation, whenever they exist, can be transformed into regular solutions by transformation of coordinates.

9 More general coordinate transformations

In section 7 we have studied coordinate transformations preserving linear differential operators with constant coefficients. Here we consider an example of a linear differential equations with non-constant coefficients. We will also begin analyzing functional coordinate transformations of nonlinear differential equations.

Example. Consider the equation (6.12) with $n = m = 1$ assuming $a(x)$ and $b(y)$ are functions. In this case the equation reads

$$a(x)\frac{\partial\omega(x,y)}{\partial x} + \frac{\partial(\omega(x,y)b(y))}{\partial y} = 0. \tag{9.1}$$

Let us look for a solution in the form

$$\omega(x,y) = e^{f(x)g(y)}. \tag{9.2}$$

Then (9.1) yields

$$a(x)f'(x)g(y) + b(y)f(x)g'(y) + b'(y) = 0. \tag{9.3}$$

If $b(y) = 1$, (9.3) is a separable equation and we have

$$\frac{a(x)f'(x)}{f(x)} = -\frac{g'(y)}{g(y)} = C, \tag{9.4}$$

where C is a constant. Solving this we obtain,

$$\omega(x,y) = e^{Ce^{\int \frac{C_1}{a(x)}dx} e^{-C_1 y}}. \tag{9.5}$$

Taking for example $C = C_1 = 1$ and $a(x) = x$, we have

$$\omega(x,y) = e^{xe^{-y}}. \tag{9.6}$$

The corresponding transformation can be shown to be injective. As we see it transforms the operator $x\delta'(y-x)$ into the operator $\delta'(y-x)$. That is,

$$\omega^{-1} : x\psi'(x) \longrightarrow \psi'(x) \tag{9.7}$$

for *any* function ψ on the space of definition of ω.

A very important question is whether we can apply the developed coordinate formalism to nonlinear differential equations. It is known that the theory of generalized functions has been mainly successful with the linear problems. The difficulty of course lies in defining the product of generalized functions. To see what kind of solution can be offered in the new context consider the following example.

Example. Consider a differential equation containing the square of the derivative of an unknown function, i.e. the term

$$\left(\varphi'(x)\right)^2, \tag{9.8}$$

where φ is an element of a Hilbert space H of functions on R. To use the coordinate formalism we need to interpret this term as a tensor. We have:

$$\varphi'(x)\cdot\varphi'(x) = \delta(x-y)\delta'(u-x)\delta'(v-y)\varphi(u)\varphi(v)dydudv, \tag{9.9}$$

where as before we omit the integral symbol. Therefore, this term is the convolution of the $(1,2)$-tensor

$$c^x_{uv} = \delta(x-y)\delta'(u-x)\delta'(v-y)dy \tag{9.10}$$

with the pair of strings $\varphi^u = \varphi(u)$. Define the function ψ by

$$c^x_{uv}\varphi^u\varphi^v = \psi^x, \tag{9.11}$$

where the meaning of notations is described in section 2. Assume that $\omega : \widetilde{H} \longrightarrow H$ is a coordinate transformation and $\omega\widetilde{\varphi} = \varphi$. Denote $\omega(x,y)$ by ω^x_y. As ψ^x is a "vector", we have

$$c^x_{uv}\omega^u_{u'}\widetilde{\varphi}^{u'}\omega^v_{v'}\widetilde{\varphi}^{v'} = \omega^x_{x'}\widetilde{\psi}^{x'}. \tag{9.12}$$

That is, the transformation properties of c^x_{uv} are as follows:

$$c^{x'}_{u'v'} = \omega^{-1}{}^{x'}_{x}\,\omega^*{}^{v}_{v'}c^x_{uv}\omega^u_{u'}, \tag{9.13}$$

where ω^* is the adjoint of ω. After "integration by parts" we obtain

$$c^x_{u'v'} = \omega^*{}^{v}_{v'}c^x_{uv}\omega^u_{u'} = \frac{\partial\omega(v',x)}{\partial x}\frac{\partial\omega(x,u')}{\partial x}. \tag{9.14}$$

By specifying the desired form of $c^{x'}_{u'v'}$ one obtains a nonlinear partial differential equation for the transformation ω. In particular, one can make φ^u, φ^v in (9.11) singular at the expense of smoothing c^x_{uv}.

Example. Consider the equation

$$\int k(x,y)\frac{d\varphi_t(x)}{dt}\frac{d\varphi_t(y)}{dt}dxdy = 0, \tag{9.15}$$

where $\varphi_t(x)$ is an unknown function which depends on the parameter t and $k(x,y)$ is a smooth function on R^{2n}.

Let us look for a solution in the form $\varphi_t(x) = \delta(x-a(t))$. As

$$\frac{d\varphi_t(x)}{dt} = -\frac{\partial\delta(x-a(t))}{\partial x^\mu}\frac{da^\mu}{dt}, \tag{9.16}$$

we have after "integration by parts" the following equation:

$$\left.\frac{\partial^2 k(x,y)}{\partial x^\mu\partial y^\nu}\right|_{x=y=a(t)}\frac{da^\mu_t}{dt}\frac{da^\nu_t}{dt} = 0. \tag{9.17}$$

If the tensor field

$$g_{\mu\nu}(a) \equiv \left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a} \tag{9.18}$$

is symmetric and positive definite, the equation (9.17) has only the trivial solution. However, if $g_{\mu\nu}(a)$ is non-degenerate and indefinite, then there is a nontrivial solution. In particular, we can choose $g_{\mu\nu}$ to be the Minkowski tensor $\eta_{\mu\nu}$ on space-time. For this assume that x, y are space-time points and take

$$k(x,y) = e^{-(x-y)^2}, \tag{9.19}$$

where $(x-y)^2 = \eta_{\mu\nu}(x-y)^\mu(x-y)^\nu$. Then it is immediate that $g_{\mu\nu}$ is the Minkowski tensor $\eta_{\mu\nu}$. Solutions to (9.17) are then given by the null lines $a(t)$. Therefore, the original problem (9.15) has solutions of the form $\varphi_t(x) = \delta(x - a(t))$ where $a(t)$ is a null line.

Notice, that the obtained generalized function φ_t is a singular generalized solution to the nonlinear equation (9.15). Once again, such a solution becomes possible because the kernel $k(x,y)$ in (9.15) is a smooth function, so the convolution $k_{xy}\dot{\varphi}^x\dot{\varphi}^y$ is meaningful.

The last two examples suggest a way of treating generalized functions in case of nonlinear equations. The results of section 8 demonstrate that a product of functions cannot be in general preserved under functional coordinate transformations. However, by treating the product as a string-tensor quantity it becomes possible to transform it to the convolution of a "good" tensor with two or more singular generalized functions. This provides one with a systematic approach to nonlinear equations in generalized functions (see [3] for further results in this direction).

10 Embedding of a finite dimensional manifold into a Hilbert space

The constructed coordinate formalism turns out to be a useful tool in formalizing the apparatus of infinite-dimensional differential geometry and in discovering yet another interesting relationship between the finite and the infinite-dimensional manifolds.

Let us first of all apply the coordinate formalism to reconsider the main ingredients of the infinite-dimensional differential geometry. For simplicity let us assume here that the Hilbert manifold S is the the string space $\mathbf{S}$ itself. Let us fix a string basis e_H on $\mathbf{S}$.

Pick a point $\Phi_0 \in \mathbf{S}$ and let $\Phi_t : R \longrightarrow \mathbf{S}$ be a differentiable path in $\mathbf{S}$ which passes through the point Φ_0 at $t = 0$. Let $\varphi = \varphi_t$ be an equation of the path in the e_H-basis (i.e. $\varphi_t = e_H^{-1}(\Phi_t)$).

The vector X *tangent* to the path Φ_t at the point Φ_0 is defined as the velocity vector of the path. In the basis e_H, X is given by

$$X = \left.\frac{d\varphi_t}{dt}\right|_{t=0}. \tag{10.1}$$

Given the vector X tangent to Φ_t at the point Φ_0 and a differentiable functional F on a neighborhood of Φ_0 in $\mathbf{S}$, the directional derivative of F at Φ_0 along X is defined by

$$XF = \left.\frac{dF(\Phi_t)}{dt}\right|_{t=0}. \tag{10.2}$$

By applying the chain rule we have:

$$XF = F'(\Phi)|_{\Phi=\Phi_0}\, \Phi'_t|_{t=0}\,, \tag{10.3}$$

where $F'(\Phi)|_{\Phi=\Phi_0} : \mathbf{S} \longrightarrow R$ is the derivative functional at $\Phi = \Phi_0$ and $\Phi'_t|_{t=0} \in \mathbf{S}$ is the derivative of Φ_t at $t = 0$.

The last expression can be also written in the coordinate form. For this notice that as $\Phi'_t|_{t=0} \in \mathbf{S}$, we also have $\varphi'_t|_{t=0} \in H$. Then

$$XF = \int \left.\frac{\delta f(\varphi)}{\delta\varphi(x)}\right|_{\varphi=\varphi_0} \xi(x)dx, \tag{10.4}$$

where $\xi = \varphi'_t|_{t=0}$ and the linear functional $\left.\frac{\delta f(\varphi)}{\delta\varphi(x)}\right|_{\varphi=\varphi_0}$, which is an element of the dual space H^*, can be thought of as the derivative functional F' in the basis e_{H^*}. As before, the integral sign is understood here in the sense of action of $\frac{\delta f(\varphi)}{\delta\varphi(x)}$ on ξ. In these notations we can also write

$$X = \int \xi(x)\frac{\delta}{\delta\varphi(x)}dx, \tag{10.5}$$

where $\xi \in H$ and the right hand side acts on functionals f defined by

$$f(\varphi) = F(\Phi), \tag{10.6}$$

where F is as before and $e_H\varphi = \Phi$. In particular, we see that in these notations tangent vectors are represented symbolically as “linear combinations” of the “partial” derivatives. Let us remark, however, that one must be careful in using this symbolic expression as on the infinite-dimensional manifold a vector field cannot be identified with a derivation (i.e. with a satisfying the product rule R-linear map acting on functions on the manifold).

The space $\mathbf{T_0S}$ of all tangent at a point Φ_0 vectors X with an appropriate topology is isomorphic to $\mathbf{S}$. In fact, it is easy to see that the map $\omega : \mathbf{T_0S} \longrightarrow \mathbf{S}$

which in the basis e_H is given by $\int \xi(x) \left.\frac{\delta}{\delta\varphi(x)}\right|_{\varphi_0} dx \longrightarrow \int e_H(x)\xi(x)dx$ is one-to-one. Therefore, ω induces a Hilbert structure on the space $\mathbf{T_0S}$. Relative to this structure ω is an isomorphism of Hilbert spaces.

The space $\mathbf{T_0S}$ with the above Hilbert structure will be called the *tangent space* to $\mathbf{S}$ at the point Φ_0. Notice that the isomorphism ω makes it possible to identify the functional basis e_H with the symbol $\frac{\delta}{\delta\varphi}$.

Let us now investigate an interesting relationship between the finite and the infinite-dimensional differential geometries in light of the coordinate formalism.

Assume we are given a specific realization of $\mathbf{S}$ as the space H of functions on R^n (or functionals of functions on R^n) . We will further assume that H contains the delta-functions $\delta(x-a)$ for various possible $a \in R^n$ (see section 3 for an example of such a realization).

Let M be the subset of H consisting of all delta-functions (without linear combinations). Notice first of all that as a subset of H, M is a topological space with the induced (subset) topology. We will further assume that the inclusion map $i : M \longrightarrow H$ is an embedding. In particular, M will be a submanifold of H.

Clearly, M is not a linear subspace of H. Nevertheless, in the simplest case the space M may have a linear structure (different than the one on H).

We will see now that the tangent bundle structure and the Riemannian structure on M are naturally induced by the embedding of M into H.

The notion of "naturality" requires some clarification. Clearly, the embedding of any finite dimensional manifold into H is always possible. Moreover, it is possible to ensure an *isometric* embedding, i.e. such that the Riemannian metric on M is a restriction (pull-back) of the Hilbert metric on H. This is due to the fact that H has "enough room" for any finite dimensional submanifold.

Since $i(M)$ consists of functions concentrated at a point (delta-functions), vectors tangent to $i(M)$ are also given by functions concentrated at a point (directional derivatives of delta-functions). Consequently, under the embedding the variational derivatives associated with vectors tangent to $\mathbf{S}$ according to (10.4), naturally reduce to the partial derivatives that can be identified with vectors tangent to M. Respectively, the induced Riemannian metric on M is obtained from the kernel $k(x, y)$ of the Hilbert metric on H by means of a *local* transformation (differentiation). The same is true about the entire algebra of tensor fields on M. As a result, the entire apparatus of the finite dimensional differential geometry on M appears as a special case of the developed coordinate formalism on infinite-dimensional manifolds.

To verify this, let us select from all paths in H the paths laying in M. For each value of the parameter t any such path φ_t reduces to a delta-function. That is,

$$\varphi_t(x) = \delta(x - a(t)) \tag{10.7}$$

for some function $a(t)$ which takes values in R^n.

It is easy to see that vectors tangent to such paths can be identified with the ordinary 4-vectors. In fact, assume f is an analytic functional on H, i.e. on a neighborhood $\|\varphi - \varphi_0\|_H < \epsilon$ of φ_0 the functional f can be represented by a uniformly convergent in a ball $\|\varphi - \varphi_0\|_{H^*} \leq \delta < \epsilon$ power series

$$f(\varphi) = f_0 + \int f_1(x)\varphi(x)dx + \int\int f_2(x,y)\varphi(x)\varphi(y)dxdy + ... \quad . \tag{10.8}$$

Then

$$\left.\frac{df(\varphi_t)}{dt}\right|_{t=0} = \left.\frac{\partial f(x)}{\partial x^\mu}\right|_{x=a(0)} \left.\frac{da^\mu}{dt}\right|_{t=0}, \tag{10.9}$$

where $f(x)$ is defined by the uniformly convergent in a ball $\|a - a_0\|_{R^4} \leq \delta_1$ in R^4 series

$$f(x) = f_0 + f_1(x) + f_2(x,x) + ... \quad . \tag{10.10}$$

In particular, the expression on the right of (10.9) can be immediately identified with the action of a 4-vector on the function $f(x)$.

To be more specific assume now that H is a real Hilbert space with the metric $K : H \times H \longrightarrow R$ given by a smooth kernel $k(x,y)$. Then the norm of a vector $\delta\varphi \in H$ can be written as

$$\|\delta\varphi\|_H^2 = \int k(x,y)\delta\varphi(x)\delta\varphi(y)dxdy. \tag{10.11}$$

If $\varphi = \varphi_t(x) = \delta(x - a(t))$ is a path in M, then $\delta\varphi(x) = \left.\frac{d\varphi_t(x)}{dt}\right|_{t=0} = -\nabla_\mu\delta(x - a)\left.\frac{da^\mu}{dt}\right|_{t=0}$, where $a = a(0)$ and derivatives are understood in a generalized sense. Therefore,

$$\|\delta\varphi\|_H^2 = \int k(x,y)\nabla_\mu\delta(x-a)\left.\frac{da^\mu}{dt}\right|_{t=0}\nabla_\nu\delta(y-a)\left.\frac{da^\nu}{dt}\right|_{t=0} dxdy. \tag{10.12}$$

"Integration by parts" in the last expression gives

$$\int k(x,y)\delta f(x)\delta f(y)dxdy = \left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a} \left.\frac{da^\mu}{dt}\right|_{t=0} \left.\frac{da^\nu}{dt}\right|_{t=0}. \tag{10.13}$$

By defining $\frac{da^\mu}{dt}|_{t=0} = dx^\mu$, we have

$$\int k(x,y)\delta f(x)\delta f(y)dxdy = g_{\mu\nu}(a)dx^\mu dx^\nu, \tag{10.14}$$

where

$$g_{\mu\nu}(a) = \left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a}. \tag{10.15}$$

As the functional K is symmetric, the tensor $g_{\mu\nu}(a)$ can be assumed symmetric as well. If in addition $\left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a}$ is positive definite at every a, the tensor $g_{\mu\nu}(a)$

can be identified with the Riemannian metric on M. Moreover, as shown in [5], [6], an arbitrary Riemannian metric on M can be locally written in the form (10.15).

In particular, consider the Hilbert space H with the metric given by the kernel $k(x, y) = e^{-(x-y)^2}$. Using (10.15) and assuming $(x - y)^2 = \delta_{\mu\nu}(x - y)^\mu (x - y)^\nu$, we immediately conclude that $g_{\mu\nu}(a) = \delta_{\mu\nu}$.

The case of a complex Hilbert space H leads in a similar way to the Kähler metric on the complex extension M^c of M. Moreover, an arbitrary Kähler metric on M^c can be obtained in such a way.

Let us also remark here that the transformations on M are in this setting just the M-preserving functional transformations on H. In particular, Lorentz transformations on Minkowski space can be identified with the special functional transformations on the appropriate Hilbert space H. The reader is referred to [6] for details.

Finally, notice that the submanifold M of H is locally parameterized by the elements $a \in R^n$ which serve as abstract parameters needed to define the elements of H. The manifold structure on M is *not* the same as on the space of parameters, but rather appears as a restriction (pull-back) of the manifold structure on H. It is therefore defined by a specific realization H^* of the abstract Hilbert space $\mathbf{S}$. After the space H and the submanifold M are chosen, the parameters a^μ can be identified with coordinates on M.

As a clarifying example consider a Hilbert space which contains the delta-functions $\delta(\theta - a)$ on R with functions $\delta(\theta - a)$ and $\delta(\theta - (a + 2\pi))$ identified for any $a \in R$. The space of parameters here is R. The space H^* is a space of generalized functions on the circle S^1. The space M consists of all delta-functions on S^1 and by the formalism of the last section is naturally identified with the unit circle itself. The parameter a becomes then identified with the angular coordinate on S^1.

11 The equation of geodesics on M

In the last section we saw how the apparatus of the finite dimensional differential geometry naturally appears as a special case of the developed functional formalism on infinite-dimensional manifolds. Here we will further extend this correspondence to demonstrate how the equation of geodesics on M can be obtained by variation of the energy functional on $\mathbf{S}$. This will also provide an example of a variational problem within the formalism.

Assume that $\mathbf{S}$ is a complex Hilbert space and let $e_H : H \longrightarrow \mathbf{S}$ be any realization of $\mathbf{S}$ as a space of functions (i.e. a functional basis on $\mathbf{S}$). The inner product on $\mathbf{S}$ can be expressed in terms of the inner product on H and will be written in one of the following ways:

$$(\varphi, \psi)_H = \int k(z, \overline{z})\varphi(z)\overline{\psi}(z)dzd\overline{z} = k_{z\overline{z}}\varphi^z\psi^{\overline{z}}. \tag{11.1}$$

As always, the integral is understood as the action of the Hermitian form K given by the kernel $k(z,\overline{z})$ and the expression on the right is a convenient form of writing this action.

Consider a curve on $\mathbf{S}$ which in the basis e_H is given by $\varphi = \varphi_t(z) \equiv \varphi_t^z$, with $t \in [a,b]$. Let us define the *square-length* (or *energy*) action functional on $\mathbf{S}$ by

$$l = \int_a^b dt k_{z\overline{z}} \frac{d\varphi_t^z}{dt} \frac{d\varphi_t^{\overline{z}}}{dt}. \tag{11.2}$$

The corresponding Lagrangian L depends only on $\frac{d\varphi}{dt} \equiv \dot{\varphi}$ and $\frac{d\overline{\varphi}}{dt} \equiv \dot{\overline{\varphi}}$, i.e. $L = L(\dot{\varphi}, \dot{\overline{\varphi}})$. For variation of l we then have:

$$\delta l = \int_a^b dt k_{z\overline{z}} \left(\ddot{\varphi}_t^z \delta \varphi_t^{\overline{z}} + \ddot{\varphi}_t^{\overline{z}} \delta \varphi_t^z \right). \tag{11.3}$$

Therefore the pair of complex conjugate equations of motion is

$$k_{z\overline{z}} \ddot{\varphi}_t^z = 0, \quad k_{z\overline{z}} \ddot{\varphi}_t^{\overline{z}} = 0. \tag{11.4}$$

If $k_{z\overline{z}}$ is an integral operator acting on integrable functions, it follows that φ_t must be a linear function of time. This is consistent with the fact that the shortest line in a Hilbert space is a straight line.

Assume that the kernel $k(z,\overline{z})$ is a smooth function. The resulting complex Hilbert space H^* contains then singular generalized functions, in particular, delta-functions. As in the previous section let us form a complex n-dimensional submanifold M^c of H^* consisting of all delta-functions in H^*. An arbitrary path in M^c is given by

$$\varphi_t(z) = \delta(z - a(t)), \quad \varphi_t(\overline{z}) = \delta(\overline{z} - \overline{a}(t)). \tag{11.5}$$

Variation of l with constraints (11.5) gives the equation (11.4) in the form

$$\int k(z,\overline{z}) \frac{d^2}{dt^2} \delta(z - a(t)) \delta\varphi_t(\overline{z}) dz d\overline{z} = 0 \tag{11.6}$$

as well as the complex conjugate equation. Notice that in a generalized sense

$$\frac{d^2}{dt^2}\delta(z-a(t)) = \frac{\partial^2}{\partial z^\nu \partial z^\mu}\delta(z-a(t))\frac{da^\nu}{dt}\frac{da^\mu}{dt} - \frac{\partial}{\partial z^\mu}\delta(z-a(t))\frac{d^2a^\mu}{dt^2}. \tag{11.7}$$

"Integration by parts" in (11.6) gives then

$$\int \left(\frac{\partial^2 k(z,\overline{z})}{\partial z^\nu \partial z^\mu} \frac{da^\nu}{dt}\frac{da^\mu}{dt} + \frac{\partial k(z,\overline{z})}{\partial z^\mu}\frac{d^2 a^\mu}{dt^2} \right) \delta(z-a(t))\delta\varphi_t(\overline{z}) dz d\overline{z} = 0. \tag{11.8}$$

Notice also that

$$\delta\varphi_t(\overline{z}) = -\frac{\partial}{\partial z^{\overline{\alpha}}}\delta(\overline{z} - \overline{a}(t))\delta a^{\overline{\alpha}}(t), \tag{11.9}$$

where $z^{\overline{\alpha}} \equiv \overline{z}^{\alpha}$ and similarly $a^{\overline{\alpha}} \equiv \overline{a}^{\alpha}$. Let us now integrate by parts with respect to $\overline{z}^{\alpha} \equiv z^{\overline{\alpha}}$ and change the order of partial derivatives. This yields

$$\int \left(\frac{\partial}{\partial z^{\mu}} \frac{\partial^2 k(z,\overline{z})}{\partial z^{\nu} \partial z^{\overline{\alpha}}} \frac{da^{\nu}}{dt} \frac{da^{\mu}}{dt} + \frac{\partial^2 k(z,\overline{z})}{\partial z^{\mu} \partial z^{\overline{\alpha}}} \frac{d^2 a^{\mu}}{dt^2} \right) \delta(z - a(t)) \delta(\overline{z} - \overline{a}) dz d\overline{z} = 0, \quad (11.10)$$

and the complex conjugate equation. Using the notation

$$g_{\mu\overline{\alpha}} = \left. \frac{\partial^2 k(z,\overline{z})}{\partial z^{\mu} \partial z^{\overline{\alpha}}} \right|_{z=a, \overline{z}=\overline{a}}, \quad (11.11)$$

we have

$$g_{\mu\overline{\alpha}} \frac{d^2 a^{\mu}}{dt^2} + \frac{\partial g_{\nu\overline{\alpha}}}{\partial z^{\mu}} \frac{da^{\nu}}{dt} \frac{da^{\mu}}{dt} = 0. \quad (11.12)$$

Assume now that the matrix $(g_{\mu\overline{\alpha}})$ is non-degenerate. Then, multiplying (11.12) by the inverse matrix $(g^{\overline{\alpha}\beta})$, we obtain

$$\frac{d^2 a^{\beta}}{dt^2} + g^{\overline{\alpha}\beta} \frac{\partial g_{\nu\overline{\alpha}}}{\partial z^{\mu}} \frac{da^{\nu}}{dt} \frac{da^{\mu}}{dt} = 0. \quad (11.13)$$

In a similar way the equation complex conjugate to (11.6) gives the complex conjugate of (11.13).

Now, it is known that any Kähler metric can be locally written in the form (11.11). Moreover, the symbols

$$\Gamma^{\beta}_{\nu\mu} = g^{\overline{\alpha}\beta} \frac{\partial g_{\nu\overline{\alpha}}}{\partial z^{\mu}} \quad (11.14)$$

are the connection coefficients for the Kähler metric (see for example [7]). That is, the equation (11.13) and the complex conjugate equation are equations of geodesics on the Kähler manifold M^c.

12 Conclusion

The passage from differentiable structure on a finite dimensional manifold to its infinite-dimensional counterpart offers an essentially new type of coordinate transformations consisting of changing Hilbert models for the manifold.

While this fact is obvious, it is rarely used because the manifold structure does not depend on a choice of the model.

Consider for example a passage from the space R^n of n-tuples to the Hilbert space l_2 of sequences. The crucial new entity on the space l_2, directly related to the infinite number of dimensions, is the notion of convergence needed to identify a sequence of numbers as an element of l_2. However, in setting up a differentiable structure on an infinite-dimensional manifold we do not pay attention to the kind of convergence available on the model space l_2. Even when, say, several Hilbert spaces of sequences are simultaneously considered as models of a Hilbert manifold, differentiable structure is not affected by the difference between them. It is concerned only with the differentiability of maps.

It is advocated here that a more productive approach to infinite-dimensional manifolds is to use the difference between Hilbert models even if the differentiable structure itself is not altered by changing a model.

The functional coordinate formalism that follows from such an approach leads to a merging of linear algebra on Hilbert spaces with the theory of generalized functions. It also provides us with an invariant functional tensor approach to linear and nonlinear problems on manifolds.

The formalism suggests that the existing approach to functional analysis overemphasizes singling out a particular functional space appropriate for a problem in hand. In the coordinate formalism such a space appears as a result of a particular (possibly, especially convenient) choice of coordinates on a manifold.

By considering several models for a manifold at once and by relating these models by isomorphisms it becomes possible to reduce the seemingly unrelated problems on different spaces of functions to a single problem on the string space. The generalized eigenvalue problem considered in section 4 provides an example.

We also obtain the possibility of reformulating a problem given on one space in terms of another space. A good example of the usefulness of such a reformulation is provided by the theory of generalized functions. In this theory operations that could not be defined directly on spaces containing singular distributions are defined first on the fundamental spaces of "good" functions. Then they are "transplanted" to the larger dual spaces. In the approach advocated here this passage from a space to its dual is a coordinate transformation and the operations themselves could be defined in an invariant manner on the string space **S**.

Moreover, all Hilbert spaces of functions including spaces of generalized functions enter the formalism on equal footing. As a result, the theory of generalized functions and linear algebra on Hilbert spaces become naturally synthesized in the formalism. It was shown in section 5 that this permits us to formulate the spectral theorem and to treat the generalized eigenvalue problems for self-adjoint operators without needing to use the rigged Hilbert space structure as in [2].

Choosing appropriate coordinates on **S** for a problem in hand is as useful (if not more) as choosing canonical coordinates for a finite dimensional problem. The simplest example of this is the Fourier transform which provides an algebraic approach to solving differential equations. Whenever useful, one can apply a coordinate change to alter analytic properties of elements of the coordinate space. The example of section 3 shows that in such a way one can transform singular distributions to infinite differentiable functions and vice versa.

We also see that in the developed approach the finite and infinite-dimensional manifolds become related in a new way. The notions of a string basis, orthogonal string basis, string eigenbasis of an operator, etc. are clear analogues of their finite dimensional counterparts. Simultaneously they provide us with the power of changing coordinate spaces and the corresponding functional description of invariant objects (tensors).

The entire approach turns out to be very similar in spirit to the nineteenth century introduction of vectors. However, it can *not* be reduced to consideration of elements of an infinite-dimensional Hilbert space as vectors. In fact, given a Hilbert space of functions elements of such space *are* vectors. The objects that we call strings are more general. They are defined for *all* Hilbert spaces of functions (i.e. coordinate spaces) at once and do not depend on a particular choice of such a space.

Preservation of various properties of operators under transformations of coordinates leads to various special types of transformations. For example, preserving locality of operators in the simplest case leads to the Fourier-like transformation as in (6.10). By finding a transformation that preserves the derivative operator we were able to relate the generalized and the regular solutions to linear partial differential equations with constant coefficients.

The results of the section 9 suggest that the nonlinear problems can be approached in the same fashion. For this it must be possible to interpret a given nonlinear equation as a tensor equation on the string space. Then various "nonlinearities" are interpreted as convolutions of tensors on the space. This seems to open a systematic way of using generalized functions in nonlinear problems.

The results of the last two sections demonstrate another unexpected relationship between the finite and the infinite-dimensional manifolds. We have seen that the coordinate formalism on the Hilbert space $\mathbf{S}$ naturally reduces to the local coordinate formalism on a (specially chosen) finite dimensional Riemannian submanifold M of $\mathbf{S}$. Moreover, under the appropriate constraints, the variational problem for a geodesic on $\mathbf{S}$ yields a geodesic on M. This seems to indicate that the finite and the infinite-dimensional manifolds can be treated together within the same local functional coordinate formalism.

The presented results provide only the first steps in applications of the formalism. A systematic approach to linear and nonlinear problems in light of the formalism is still lacking. It is the author's hope that his interest in the formalism and its applications will be shared and that a complete understanding of the formalism will be eventually obtained.

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