On the problem of emergence of classical spacetime: The quantum-mechanical approach

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The Riemannian manifold structure of the classical (i.e. Einsteinian) spacetime is derived from the structure of an abstract infinite-dimensional separable Hilbert space **S**. For this **S** is first realized as a Hilbert space H of functions of abstract parameters. The space H is associated with the space of states of a macroscopic test-particle in the universe. The spatial localization of state of the particle through its interaction with the environment is associated with the selection of a submanifold M of realization H. The submanifold M is then identified with the classical space (i.e. a space-like hypersurface in spacetime). The mathematical formalism is developed which allows recovering of the usual Riemannian geometry on the classical space and, more generally, on space and time from the Hilbert structure on **S**. The specific functional realizations of **S** are capable of generating spacetimes of different geometry and topology. Variation of the length-type action functional on **S** is shown to produce both the equation of geodesics on M for macroscopic particles and the Schrödinger equation for microscopic particles.

Key words: spacetime, emergence, Hilbert manifolds, generalized functions

1 Introduction

We take for granted that physical events take place in spacetime. Mathematically this is reflected in realization of physical quantities as functions (in a broad sense) of spacetime points. The shortcomings of such a realization are well known. In particular, the position of a particle in quantum field theory (QFT) is only defined to energies less than the particle's mass. The concept of a field at a point is ill-defined as well.

By now the string/M-theory is believed by many to be the leading successor of the QFT. Not only does it deal successfully with the divergences plaguing QFT, but it also leads to a unified approach to the known interactions. However, one of the main objections to the string/M-theory is that it requires the notion of spacetime to begin with. In particular, the theory, while deducing gravity, does not deduce

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the spacetime which therefore remains classical. As Brian Green puts it in Ref.[1]: "Finding the correct mathematical apparatus for formulating string theory without recourse to a pre-existing notion of space and time is one of the most important issues facing string theorists. An understanding of how the space and time emerge would take us a huge step closer to answering the crucial question of which geometric form actually *does* emerge."

The situation in string/M-theory is reminiscent of non-relativistic quantum mechanics (QM). While the former theory requires the classical background for its existence, the latter seems to require the classical behavior of the measuring devices. The fundamental problem of deducing the classical world from the quantum one is therefore common to both theories.

The concept of spacetime in the classical theory is intimately related to the material content of spacetime. In fact, in the theory of gravity matter together with the initial data determines the Riemannian structure of spacetime. The Riemannian, or, more precisely, the pseudo-Riemannian structure will be also called the *macroscopic structure* of spacetime. A spacetime with such a structure will be called *classical*.

By contrast, the *microscopic structure* of spacetime is expected to be determined (and, in fact, defined) by the quantum theory. The microscopic structure, although currently unknown, must also depend on the state of matter in spacetime. This follows already from the expected transition to the macroscopic structure, in accordance with the correspondence principle.

The localization of space wave packets is essential in bridging the gap between the "macro" and the "micro" structures of spacetime in the QM approximation. In fact, to physically (i.e. experimentally) determine the macroscopic structure one must be able to identify points of spacetime. To physically identify a point is to observe an event at the point. Typically, observations of this kind are done by means of scattering experiments. As a result of a high resolution scattering event needed to identify the point, the scattering center, i.e. a particle, assumes a *localized* state. That is, within the QM approximation the state function of the particle at the moment of interaction is a wave packet localized in space.

In a hypothetical situation, where no spatially localized wave packets are present in a region Ω of spacetime, it becomes meaningless to refer in a *physical* way to a particular point of Ω. In fact, no event at a point of Ω is then observed. As a result, the Riemannian structure itself becomes unobservable.

We are then faced with the following three options. One could continue insisting that the region Ω has underlying structure of a four-dimensional Riemannian manifold, although unobservable. Or else, one could say that the region has no manifold structure at all. Finally, one could argue that the geometry in Ω is a new, "quantum" geometry.

If the last option (investigated here) is chosen, it is important to be able to "probe" the new geometry experimentally so as to avoid making it ad hoc. That means that the new geometry, in the QM approximation (provided such an approximation exists) must be "encoded" in the state functions or density matrices of systems of particles in Ω.

For simplicity, assume that the system consists of a single particle in a pure state. In the QM formalism both localized and non-localized states of the particle are elements of a Hilbert space of states H. What distinguishes the localized states is that they determine the region, (ideally, the point) of localization. Such a region is the support of the state function. On the other hand, if the position of a particle at a moment of time is known, the state function of the particle at this moment is assumed in QM to be the Dirac delta-function.

It follows that the points of space are in one-to-one correspondence with the states of a particle localized at these points. It seems therefore reasonable to identify the *classical space*, i.e. a space-like hypersurface in spacetime, with the subset $M_3 \subset H$ consisting of all point-supported state functions in H. For an appropriately chosen space H such a subset will be shown to be a Riemannian submanifold of dimension three with the metric induced by inclusion. As a superposition of delta-functions cannot be in general reduced to a delta-function, the submanifold M_3 is not a linear subspace of H. The latter fact may be considered as a trait of classicality of M_3. It will be shown that M_3 may still possess a linear structure, however incompatible with the one on H.

Of course the embedding, and even the assumed here isometric embedding of a finite dimensional Riemannian manifold into a Hilbert space is always possible and may seem rather trivial and artificial. There is, however, an important circumstance which supports the proposed embedding. Namely, the macroscopic particles (i.e. those having a sufficiently large mass) in the universe *are* found in spatially well localized space wave packets. Simultaneously, sufficiently small in size and mass macroscopic particles are the ones that can be used as test-particles to "probe" the Riemannian manifold structure of the classical spacetime. It follows that the proposed embedding appears *naturally* in the universe. It is then plausible that within the formalism of QM the origin of the classical space can be traced back to the Hilbert space of states of a *macroscopic test-particle*. The latter is understood as a material point of a large enough mass to behave classically but small enough to affect the Riemannian structure of spacetime. In the following such a particle will be simply called a *test-particle*.

Although the actual mechanism of localization will not be important in the following, let us briefly review the most typical scenario of spatial localization of a test-particle. A test-particle in the universe is subject to constant scattering processes. A scattering process depends on position of the particle. That is, the scattered photons and other particles (i.e. the *environment*) "measure" position of the test-particle. Assume that prior to a scattering event the test-particle is in the state φ_i corresponding to the particle being at a particular location a_i. Then the state of the composite particle-environment system after the interaction can be described by the direct product $\varphi_i \Phi_i$, where Φ_i is the (unitary evolved) state of the environment which contains now information about location of the test-particle. If instead, prior to measurement the test-particle is in a superposition $c_1\varphi_1 + c_2\varphi_2$ of states of the particle being "here and there", the linearity of QM ensures that the composite system after the interaction will be in the state $\Psi = c_1\varphi_1\Phi_1 + c_2\varphi_2\Phi_2$. As a result, the states of the particle and the environment become essentially correlated or *entangled*.

However, when the number of degrees of freedom associated with the environment is large, the states Φ_i for different values of i turn out to be orthogonal. In fact, in this case the inner product of Φ_i's is the product of a large number of factors in general of magnitude less then one and is therefore exponentially small. As a result, interference effects between the states φ_1, φ_2 vanish. The resulting physical process of transition of a pure state $c_1\varphi_1 + c_2\varphi_2$ into the mixture of states φ_1, φ_2 with probabilities $|c_1|^2, |c_2|^2$ respectively is called *decoherence*.

The mixture of states φ_1, φ_2 is not, however, what is observed when position of a test-particle in QM is measured. Instead, we observe only one of the pure states in the mixture. The selection of a particular term out of the mixture remains, therefore, unexplained (see in particular Ref.[4]). The process of such a selection is called *collapse* and it is currently a subject under intensive investigation. In particular, various stochastic extensions of the Schrödinger dynamics leading to decoherence and collapse have been proposed (see in particular Refs.[5],[6]).

Whatever the actual mechanism of decoherence and collapse of a state of macroscopic test-particle in the universe may be, it must ensure the observed spatial localization of the particle. This leads us back to the conclusion that the classical space can be identified with the set of point-supported state functions of the test-particle. The space M_3 therefore appears as a "decohered and collapsed" version of the Hilbert space H of states of the particle.

Besides being physically motivated, the isometric embedding of M_3 into H turns out to be special in a mathematical sense. In fact, we will see that such an embedding allows one to derive the standard local coordinate formalism of differential geometry on Riemannian manifolds from the functional coordinate formalism on Hilbert manifolds introduced in Ref.[7] (see also section 2 of this paper). It will follow that the proposed embedding is by no means arbitrary or trivial.

Even though the localized states can be identified with the points of classical space in the described natural way, one can argue that the resulting embedding of the space M_3 into a Hilbert space of states H achieves nothing. In fact, instead of the advance toward understanding of emergence of space and time, we consider functions *on* the space to begin with.

This objection, however, is unwarranted: The identification of points with the states makes the assumption that the state functions are defined *on* the classical space superfluous. In fact, the space M_3 does not appear as a set on which the functions are defined. Instead, M_3 itself is a set of functions in H selected in a particular way. In other words, the space M_3 is "made of" functions and not of points in the domain of the functions. In particular, the elements in H may be assumed to be functions of abstract parameters, where parameters have nothing to do with the points in the classical space. Instead, they only serve to identify functions as particular elements of the Hilbert space H. The space of parameters in the paper will be either three or four dimensional Euclidean space E. The secondary role of parameters is reflected in the fact that topologically and/or metrically different spaces M_3 can be obtained by considering various Hilbert spaces H of functions defined *on the same set* E. The reason for that is simple: the topology and the metric on M_3 depend only on the embedding of M_3 into H and have nothing to do with the actual nature of the elements of H. The reader is referred to section 4 for a thorough examination of this fact.

We conclude that the proposed scenario leads one to a model of emergence of the classical space. Namely, it is hypothesized that an abstract Hilbert space $\mathbf{S}$ is a new physical arena which replaces the classical space. The emergence of the latter in the model is associated with a physical process of "localization" by self-interaction of the universe. Such a process must be independent of the classical space dynamical process on $\mathbf{S}$.

Creating a model according to the provided scenario is a very ambitious task which cannot be undertaken in the paper. The goal that we have in mind here is much more modest and consists in finding the mathematical formalism appropriate for the model. In particular, we leave out the details of dynamics leading to emergence of the classical space and to its embedding into a Hilbert space. Instead, we concentrate on the properties of embedding itself and on the affect of embedding on dynamics of macroscopic and microscopic particles.

Here is a plan of the paper. In the next section we develop a mathematical formalism (the "embedding formalism") that reveals a simple and useful relationship between the finite and the infinite-dimensional manifolds. The formalism is a natural application of the coordinate formalism on Hilbert manifolds developed earlier in Ref.[7]. To keep the exposition self-contained the results of Ref.[7] used in the paper are briefly reviewed.

In section 3 the geometry of embedding is used to derive the equation of geodesics for macroscopic particles and the Schödinger equation for microscopic particles. Both equations are derived by variation of a length-type functional on paths in an appropriate Hilbert space. This result indicates that the formalism may be rich enough to describe the macro- and the micro-reality in a uniform fashion.

The results are summarized, extended and further clarified in the last part of the paper. Here we explain in detail why does it become unnecessary in the model to presuppose the existence of the classical space. It is shown that, on the contrary, various topologically different classical spaces can be derived by a "coordinate transformation" on the abstract Hilbert space. The main results of the paper are further analyzed in this section and the steps needed to make the model realistic are discussed.

Let us remark that the provided "emergence scenario" is based on the formalism of QM. Therefore, the approach is non-relativistic in nature. In particular, time plays a distinguished role in the discussion. As a result, the model under investigation is a model of emergence of the classical space rather than of space *and* time. The formalism itself can be easily extended to describe the embedding of space-time into a Hilbert space. We will use this fact in the paper to obtain a formally relativistic embedding. However, the Hilbert space in this case will be a space of functions of four variables. Respectively, the meaning of the formalism warrants further investigation.

Let us also remark that the Hilbert space under consideration is the space of possible states of a *single* test-particle. This will be sufficient to clarify the formalism. Moreover, the developed formalism could be generalized to include Hilbert spaces of states of more complicated quantum systems, for example, of a system of non-interacting test-particles. However, such generalizations are not considered in the paper.

2 The embedding formalism

Let **S** be the abstract infinite-dimensional separable Hilbert space. Let H be a specific functional realization of **S** as a Hilbert space of states of a macroscopic test-particle in the universe. Ideally such a space H must contain delta-functions. In fact, as discussed in the previous section, the macroscopic particles in the universe are found in well localized space wave packets. It is commonly believed that the delta-like states in QM cannot be elements of a Hilbert space. The existence of various Hilbert spaces of generalized functions shows that this opinion is wrong. Consider for example the Sobolev space $H \equiv H^1(a,b)$ of functions on the interval $[a,b] \subset R$, where R is the set of real numbers, with the inner product $(\varphi,\psi)_H = \int (d\varphi/dx d\psi/dx + \varphi\psi)dx$. This is a separable Hilbert space embedded into the space $C[a,b]$ of continuous functions on $[a,b]$ (by the Sobolev embedding theorem). The dual Hilbert space H^* contains then as a subset the space of linear continuous functionals on $C[a,b]$. For example, H^* contains the delta functional which has therefore a finite norm in H^*. It follows in particular that H is a proper subset of H^*, where we identify regular functionals with the corresponding functions. At the same time, by the Riesz theorem, H is isomorphic to H^* and any functional $f \in H^*$ can be written in the form $f(\psi) = (\varphi,\psi)_H$ for some $\varphi \in H$. In particular, the delta-functional can be written in such a way for a continuous function φ. The resulting functional is singular not because of the singularity of φ, but because the metric $\widehat{G} : H \longrightarrow H^*$ on H transforms φ into a singular generalized function.

Moreover, the coordinate formalism of Ref.[7] permits one to consider Hilbert spaces containing singular generalized functions as well as spaces of square-integrable functions on an equal footing. For this the metric on Hilbert spaces of functions is made dependent (in a "covariant" fashion) on the variety of functions making up a particular space. Namely, consider a Hilbert space H of functions finite in the metric associated with the inner product

$$(\varphi, \psi)_H = \int k(x,y)\varphi(x)\psi(y)dxdy. \tag{2.1}$$

In Eq.(2.1) the kernel $k(x,y)$ is an appropriate function on, say, $R^n \times R^n$ and the integral sign is understood as the action of the corresponding bilinear functional on $H \times H$ (see Ref.[7] for notation). More constructively, H can be obtained by completing a space of ordinary functions φ with respect to the norm $||\varphi||_H^2 = (\varphi, \varphi)_H$. We remark here that only those functions $k(x,y)$ for which Eq.(2.1) is a non-degenerate inner product (i.e. the corresponding completion H is a Hilbert space) are considered.

By changing the "smoothness" properties of $k(x,y)$ as well as its behavior at infinity we change the variety of functions in H. If, for example, the kernel $k(x,y)$ is a smooth function, then the corresponding Hilbert space contains various singular generalized functions.

In particular, the space H of real valued generalized functions "of" (i.e. defined on functions of) $x \in R^n$ finite in the metric

$$(\varphi, \psi)_H = \int e^{-(x-y)^2}\varphi(x)\psi(y)dxdy \tag{2.2}$$

can be shown to be Hilbert (see Ref.[7]). Such a space contains the delta-functions as, for example,

$$\int e^{-(x-y)^2}\delta(x)\delta(y)dxdy = 1. \tag{2.3}$$

Moreover, H contains the derivatives of any order of the delta-functions as well.

Throughout the rest of this section Hilbert spaces with the metric defined by a *smooth* kernel $k(x,y)$ will be generically denoted by H. Given a particular such realization H of $\mathbf{S}$, consider a submanifold M of H consisting only of delta-functions (in particular, the superpositions of delta-functions are discarded). The fact that M is a submanifold of H follows immediately from the easily verified differentiability of the parametrization map $P : R^n \longrightarrow H$, $P(a) = \delta(x-a)$. As follows from the definition, M is not a linear subspace of H. However, if the parametrization map P is defined on the entire R^n, it induces a linear structure on M (incompatible with the one on H).

Let us relate the differential geometry of abstract Hilbert space $\mathbf{S}$ and its realization H with the usual differential geometry on M. Let $e_H : H \longrightarrow \mathbf{S}$ be a particular realization of $\mathbf{S}$ as a Hilbert space of functions. In other words, e_H is a functional basis on $\mathbf{S}$ (see Ref.[7]). Pick a point $\Phi_0 \in \mathbf{S}$ and let $\Phi_t : R \longrightarrow \mathbf{S}$ be a differentiable path in $\mathbf{S}$ which passes through the point Φ_0 at $t=0$.

The vector X *tangent* to the path Φ_t at the point Φ_0 is defined as the velocity vector of the path:

$$X = \left.\frac{d\Phi_t}{dt}\right|_{t=0}. \tag{2.4}$$

Given vector X tangent to Φ_t at the point Φ_0 and a differentiable functional F on a neighborhood of Φ_0 in $\mathbf{S}$, the directional derivative of F at Φ_0 along X is defined by

$$XF = \left.\frac{dF(\Phi_t)}{dt}\right|_{t=0}. \tag{2.5}$$

By applying the chain rule we have:

$$XF = F'(\Phi)|_{\Phi=\Phi_0}\, \Phi_t'|_{t=0}, \tag{2.6}$$

where $F'(\Phi)|_{\Phi=\Phi_0} : \mathbf{S} \longrightarrow R$ is the derivative functional at $\Phi = \Phi_0$.

The last expression can be also written in the coordinate form. Namely,

$$XF = \int \left.\frac{\delta f(\varphi)}{\delta\varphi(x)}\right|_{\varphi=\varphi_0} \xi(x)dx, \tag{2.7}$$

where $\varphi_t = e_H^{-1}(\Phi_t)$, $\xi = \varphi_t'|_{t=0}$ and the linear functional $\left.\frac{\delta f(\varphi)}{\delta\varphi(x)}\right|_{\varphi=\varphi_0}$, which is an element of the dual space H^*, can be thought of as the derivative functional $F'(\Phi_0)$ in the dual basis e_H^* (see Ref.[7]). As before, the integral sign is understood here in the sense of action of $\frac{\delta f(\varphi)}{\delta\varphi(x)}$ on ξ. In this notation we can also write

$$X = \int \xi(x)\frac{\delta}{\delta\varphi(x)}dx, \tag{2.8}$$

where $\xi \in H$ and the right hand side acts on functionals f defined by

$$f(\varphi) = F(\Phi), \tag{2.9}$$

where F is as before and $e_H\varphi = \Phi$. In particular, we see that in this notation tangent vectors are represented symbolically as "linear combinations" of the "partial" derivatives. Let us remark, however, that one must be careful in using this symbolic expression as on the infinite-dimensional manifolds a vector field cannot be identified with a *derivation* (i.e. an R-linear map defined on functions on a manifold and satisfying the product rule).

Recall that M denotes the submanifold of H consisting of all delta-functions (without linear combinations). Let us see how the tangent bundle structure and the Riemannian structure on M are induced by the embedding $i : M \longrightarrow H$. For this, let us select from all paths in H the paths taking values in M. Any such path $\varphi_t : R \longrightarrow M$ can be defined by

$$\varphi_t(x) = \delta(x - a(t)) \tag{2.10}$$

for some function $a(t)$ which takes values in R^n.

It is easy to see that vectors tangent to such paths can be identified with the ordinary 4-vectors. In fact, assume f is an analytic functional on H, i.e. on a neighborhood $\|\varphi - \varphi_0\|_H < \epsilon$ of φ_0 the functional f can be represented by a uniformly convergent in a ball $\|\varphi - \varphi_0\|_H \leq \delta < \epsilon$ series

$$f(\varphi) = f_0 + \int f_1(x)\varphi(x)dx + \int\int f_2(x,y)\varphi(x)\varphi(y)dxdy + ... \quad . \tag{2.11}$$

Then

$$\left.\frac{df(\varphi_t)}{dt}\right|_{t=0} = \left.\frac{\partial f(x)}{\partial x^\mu}\right|_{x=a(0)} \left.\frac{da^\mu}{dt}\right|_{t=0}, \tag{2.12}$$

where $f(x)$ is defined by the uniformly convergent in a ball $\|a - a_0\|_{R^n} \leq \delta_1$ in R^n series

$$f(x) = f_0 + f_1(x) + f_2(x,x) + ... \quad . \tag{2.13}$$

In particular, the expression on the right of Eq.(2.12) can be immediately identified with the action of a n-vector $\frac{da^\mu}{dt}\frac{\partial}{\partial x^\mu}$ on the function $f(x)$.

As before, let H be a real Hilbert space with the metric $K : H \times H \longrightarrow R$ given by a smooth kernel $k(x,y)$. The norm of a vector $\delta\varphi \in H$ can be written as

$$\|\delta\varphi\|^2_H = \int k(x,y)\delta\varphi(x)\delta\varphi(y)dxdy. \tag{2.14}$$

If $\varphi_t(x)$ is a path with values in M, then $\left.\frac{d\varphi_t(x)}{dt}\right|_{t=0} = -\nabla_\mu\delta(x-a)\left.\frac{da^\mu}{dt}\right|_{t=0}$, where $a = a(0)$, $\nabla_\mu = \frac{\partial}{\partial x^\mu}$ and derivatives are understood in a generalized sense. Therefore, for $\delta\varphi(x) = \left.\frac{d\varphi_t(x)}{dt}\right|_{t=0}$ we have:

$$\|\delta\varphi\|^2_H = \int k(x,y)\nabla_\mu\delta(x-a)\left.\frac{da^\mu}{dt}\right|_{t=0}\nabla_\nu\delta(y-a)\left.\frac{da^\nu}{dt}\right|_{t=0}dxdy. \tag{2.15}$$

"Integration by parts" in the last expression gives

$$\int k(x,y)\delta\varphi(x)\delta\varphi(y)dxdy = \left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a}\left.\frac{da^\mu}{dt}\right|_{t=0}\left.\frac{da^\nu}{dt}\right|_{t=0}. \tag{2.16}$$

By defining $\frac{da^\mu}{dt}|_{t=0} = dx^\mu$, we have

$$\int k(x,y)\delta\varphi(x)\delta\varphi(y)dxdy = g_{\mu\nu}(a)dx^\mu dx^\nu, \tag{2.17}$$

where

$$g_{\mu\nu}(a) = \left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a}. \tag{2.18}$$

As the functional K is symmetric, the tensor $g_{\mu\nu}(a)$ can be assumed symmetric as well. If in addition $\left.\frac{\partial^2 k(x,y)}{\partial x^\mu \partial y^\nu}\right|_{x=y=a}$ is positive definite at every a, the tensor $g_{\mu\nu}(a)$ can be identified with the Riemannian metric on M.

In particular, consider the Hilbert space H with the metric given by the kernel $k(x,y) = e^{-\frac{1}{2}(x-y)^2}$. Using Eq.(2.18) and assuming $(x-y)^2 = \delta_{\mu\nu}(x-y)^{\mu}(x-y)^{\nu}$, we immediately conclude that $g_{\mu\nu}(a) = \delta_{\mu\nu}$.

If instead $(x-y)^2$ is used for $\eta_{\mu\nu}(x-y)^{\mu}(x-y)^{\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric on R^n, we obtain $g_{\mu\nu}(a) = \eta_{\mu\nu}$. In particular both, the Euclidean and the Minkowski metrics on M can be obtained from a metric on H. Notice that the kernel $e^{-\frac{1}{2}\eta_{\mu\nu}(x-y)^{\mu}(x-y)^{\nu}}$ does not define a positive-definite bilinear functional on $H \times H$. However, the functional is non-degenerate.

These examples show that the Euclidean and Minkowski spaces can be obtained as submanifolds M consisting of delta-functions in a Hilbert space H. It is important to know, however, whether an arbitrary Riemannian metric $g_{\mu\nu}(a)$ can be obtained in such a way. That is, given a manifold N with a Riemannian metric g does there exist a Hilbert space H, such that the submanifold M consisting of delta-functions in H with the induced metric is, at least locally, isometric to N?

Clearly, for a function $k(x)$ with $x \in R^n$ the tensor field $\frac{\partial^2 k(x)}{\partial x^{\mu}\partial x^{\nu}}$ is rather special and cannot be made equal to an arbitrary Riemannian metric on R^n. However, we have twice as many variables at our disposal. To analyze the situation assume for a moment that the space H is a complex Hilbert space of (generalized) functions "on" C^n. The variables x, y in Eq.(2.14) are then replaced with the complex conjugate variables $z, \overline{z}$. The Hilbert metric on H is necessary Hermitian. This can be assured, in particular, by choosing a real-valued kernel $k(z,\overline{z})$. In this case the form $g_{\mu\overline{\nu}} = \frac{\partial^2 k(z,\overline{z})}{\partial z^{\mu}\partial \overline{z}^{\nu}}$ is automatically Hermitian as well. If, in addition, $g_{\mu\overline{\nu}}$ is positive definite, the Riemannian metric $g_{\mu\overline{\nu}}$ is known to be Kähler. Moreover, an arbitrary Kähler metric on C^4 can be written locally in such a way (see for example Ref.[2]). Therefore, for Kähler manifolds the answer to the above question is positive, provided the Kähler potential $k(z,\overline{z})$ defines a Hilbert metric.

A similar result holds true for an arbitrary real analytic Riemannian manifold. In fact, in Ref.[8] it was verified that *any* real analytic Riemannian n-dimensional manifold can be real analytically and isometrically embedded into a Kähler n-dimensional manifold. Then the previous consideration can be applied. Moreover, this result demonstrates that the complex Hilbert space structure on H naturally leads one to a Kähler structure on the complex extension of spacetime. The results of Ref.[8] allow one to make similar statements in case of pseudo-Riemannian manifolds.

Let us review what makes the embedding $i : M \longrightarrow H$ special. Since $i(M)$ consists of functions concentrated at a point (delta-functions), vectors tangent to $i(M)$ are also given by functions concentrated at a point (directional derivatives of delta-functions). Consequently, under the embedding the variational derivatives associated with vectors tangent to $\mathbf{S}$ according to Eq.(2.7), naturally reduce to the partial derivatives that can be identified with vectors tangent to M. Respectively, the induced Riemannian metric on M is related to the kernel $k(x,y)$ of the Hilbert metric on H by means of a *local* transformation (differentiation). As a result, the standard local coordinate formalism on M appears as a special case of the coordinate formalism on infinite-dimensional manifolds presented in Ref.[7]. Moreover, the obtained results indicate how a finite dimensional Riemannian manifold N can be isometrically embedded into a Hilbert space H as a submanifold consisting of delta-functions. Such an embedding allows one to consider the finite and the infinite-dimensional manifolds within the same local coordinate formalism. This gives us a reason to call the embedding $i : M \longrightarrow H$ *natural*.

Let us now turn to some applications of the embedding formalism.

3 Variational problems for "macro" and "micro" particles

In the last section we saw how the formalism of finite dimensional Riemannian geometry in local coordinates is naturally derived from the coordinate formalism on Hilbert manifolds developed in Ref.[7]. As a first application of the embedding formalism we will obtain the equation of geodesics on M by variation of a functional on paths in $\mathbf{S}$. This is particularly interesting as it demonstrates how the exceptional properties of the embedding $i : M \longrightarrow H$ allows one to deduce the equation of geodesics on M by "localizing" to M the equation of geodesics on $\mathbf{S}$.

Example. Assume that $\mathbf{S}$ is a complex Hilbert space and let $e_H : H \longrightarrow \mathbf{S}$ be a functional basis on $\mathbf{S}$, i.e. any realization of $\mathbf{S}$ as a space of complex-valued functions on C^n. The inner product on $\mathbf{S}$ can be expressed in terms of the inner product on H and will be written in one of the following ways:

$$(\varphi, \psi)_H = \int k(z, \overline{z})\varphi(z)\overline{\psi}(z)dzd\overline{z} = k_{z\overline{z}}\varphi^z\psi^{\overline{z}}. \tag{3.1}$$

As always, the integral is understood as the action of the Hermitian form K given by the kernel $k(z, \overline{z})$ and the expression on the right is a convenient form of writing this action.

Consider a path on $\mathbf{S}$ which in the basis e_H is given by a map $\varphi : t \longrightarrow \varphi_t(z) \equiv \varphi_t^z$, where t takes values in some interval $[a, b]$ of real numbers. Let us define the *square-length* (or *energy*) action functional on paths by

$$l(\varphi) = \int_a^b dt k_{z\overline{z}} \frac{d\varphi_t^z}{dt}\frac{d\varphi_t^{\overline{z}}}{dt}. \tag{3.2}$$

The corresponding Lagrangian L depends only on $\frac{d\varphi}{dt} \equiv \dot{\varphi}$ and $\frac{d\overline{\varphi}}{dt} \equiv \dot{\overline{\varphi}}$, i.e. $L = L(\dot{\varphi}, \dot{\overline{\varphi}})$. For variation of $l(\varphi)$ we then have:

$$\delta l(\varphi) = \int_a^b dt k_{z\overline{z}} \left(\ddot{\varphi}_t^z \delta\varphi_t^{\overline{z}} + \ddot{\varphi}_t^{\overline{z}}\delta\varphi_t^z\right). \tag{3.3}$$

Therefore, the pair of complex conjugate equations of motion is

$$k_{z\overline{z}}\ddot{\varphi}_t^z = 0, \quad k_{z\overline{z}}\ddot{\varphi}_t^{\overline{z}} = 0. \tag{3.4}$$

Since the Hilbert metric is non-degenerate, it follows that $\ddot{\varphi}_t^z = 0$, i.e. φ_t is a linear function of the parameter t. This is consistent with the fact that the shortest line in a Hilbert space is a straight line.

Assume that the kernel $k(z, \overline{z})$ is a smooth function. The resulting complex Hilbert space H contains then singular generalized functions, in particular, delta-functions. Analogously to the previous section, let us form a complex n-dimensional submanifold M^c of H consisting of all delta-functions in H. An arbitrary path in M^c is given by

$$\varphi_t(z) = \delta(z - a(t)), \quad \varphi_t(\overline{z}) = \delta(\overline{z} - \overline{a}(t)). \tag{3.5}$$

Variation of l with constraints Eq.(3.5) yields

$$\int k(z, \overline{z}) \frac{d^2}{dt^2} \delta(z - a(t)) \delta\varphi_t(\overline{z}) dz d\overline{z} = 0 \tag{3.6}$$

as well as the complex conjugate equation. Here the variation $\delta\varphi_t(\overline{z})$ must respect Eq.(3.5). Notice that in a generalized sense

$$\frac{d^2}{dt^2} \delta(z - a(t)) = \frac{\partial^2}{\partial z^\nu \partial z^\mu} \delta(z - a(t)) \frac{da^\nu}{dt} \frac{da^\mu}{dt} - \frac{\partial}{\partial z^\mu} \delta(z - a(t)) \frac{d^2 a^\mu}{dt^2}. \tag{3.7}$$

"Integration by parts" in Eq.(3.6) gives then

$$\int \left(\frac{\partial^2 k(z, \overline{z})}{\partial z^\nu \partial z^\mu} \frac{da^\nu}{dt} \frac{da^\mu}{dt} + \frac{\partial k(z, \overline{z})}{\partial z^\mu} \frac{d^2 a^\mu}{dt^2} \right) \delta(z - a(t)) \delta\varphi_t(\overline{z}) dz d\overline{z} = 0. \tag{3.8}$$

Notice also that

$$\delta\varphi_t(\overline{z}) = -\frac{\partial}{\partial z^{\overline{\alpha}}} \delta(\overline{z} - \overline{a}(t)) \delta a^{\overline{\alpha}}(t), \tag{3.9}$$

where $z^{\overline{\alpha}} \equiv \overline{z}^\alpha$ and similarly $a^{\overline{\alpha}} \equiv \overline{a}^\alpha$. Let us now integrate by parts with respect to $z^{\overline{\alpha}}$ and change the order of partial derivatives. This yields

$$\int \left(\frac{\partial}{\partial z^\mu} \frac{\partial^2 k(z, \overline{z})}{\partial z^\nu \partial z^{\overline{\alpha}}} \frac{da^\nu}{dt} \frac{da^\mu}{dt} + \frac{\partial^2 k(z, \overline{z})}{\partial z^\mu \partial z^{\overline{\alpha}}} \frac{d^2 a^\mu}{dt^2} \right) \delta(z - a(t)) \delta(\overline{z} - \overline{a}) dz d\overline{z} = 0, \tag{3.10}$$

and the complex conjugate equation. Using the notation

$$g_{\mu\overline{\alpha}}(a) = \left. \frac{\partial^2 k(z, \overline{z})}{\partial z^\mu \partial z^{\overline{\alpha}}} \right|_{z=a, \overline{z}=\overline{a}}, \tag{3.11}$$

we have

$$g_{\mu\overline{\alpha}} \frac{d^2 a^\mu}{dt^2} + \frac{\partial g_{\nu\overline{\alpha}}}{\partial z^\mu} \frac{da^\nu}{dt} \frac{da^\mu}{dt} = 0. \tag{3.12}$$

Assume now that the matrix $(g_{\mu\overline{\alpha}})$ is non-degenerate. Then, multiplying Eq.(3.12) by the inverse matrix $(g^{\overline{\alpha}\beta})$, we obtain

$$\frac{d^2 a^\beta}{dt^2} + g^{\overline{\alpha}\beta} \frac{\partial g_{\nu\overline{\alpha}}}{\partial z^\mu} \frac{da^\nu}{dt} \frac{da^\mu}{dt} = 0. \tag{3.13}$$

In a similar way the equation complex conjugate to Eq.(3.6) gives the complex conjugate of Eq.(3.13).

Now, as discussed at the end of section 2, any Kähler metric can be locally written in the form Eq.(3.11). Moreover, the symbols

$$\Gamma^{\beta}_{\nu\mu} = g^{\overline{\alpha}\beta}\frac{\partial g_{\nu\overline{\alpha}}}{\partial z^{\mu}} \tag{3.14}$$

are the connection coefficients for the Kähler metric Eq.(3.11) (see for example Ref.[2]). That is, the equation Eq.(3.13) and the complex conjugate equation are equations of geodesics on the Kähler manifold M^c.

Also, any real analytic Riemannian n-dimensional manifold M can be isometrically embedded into a Kähler manifold M^c of complex dimension n. Moreover, one can always assure M to be a totally geodesic submanifold of M^c (see Ref.[8]). This means that every geodesic on M is also a geodesic on M^c. In particular, geodesics of M, when considered as paths in M^c satisfy Eq.(3.13).

This demonstrates that the extremals of the functional $l(\varphi)$ in Eq.(3.2) subject to the constraint $\varphi_t(x) = \delta(x - a(t))$ yield geodesics on M. Assume in particular that H is a Hilbert space of functions of four abstract variables and, as before, M is a submanifold made of delta-functions in H. Assume further that the induced metric on M is pseudo-Riemannian and M is identified with the spacetime. Notice that this requires the metric on H to be an indefinite Hilbert metric. Then the obtained result means that the dynamics of test-particles in a field of gravity can be deduced by variation of the square-length functional on paths in H. In addition, as explained in detail in section 4, this derivation does not presuppose the existence of the classical space. Notice however, that in this case the elements of H, being functions of four variables, cannot be directly interpreted as quantum states. As already mentioned in the introduction, this fact warrants further investigation.

In the next example we consider the microscopic particles instead.

Example. Assume that a microscopic particle is in a stationary state so that

$$\varphi_t(x) = e^{-i\omega t}\psi(x), \tag{3.15}$$

where x in Eq.(3.15) refers to coordinates in space and t is time. Respectively, let H be a realization of $\mathbf{S}$ as a space of complex-valued functions ψ of three abstract variables for which we still use the notation x. As we explain in section 4, neither the derivation nor its interpretation require functions ψ to be defined on the classical space.

For any $\psi \in H$ the function $\varphi_t : R \longrightarrow H$ defined by Eq.(3.15) is a path with values in H. As the variables x are real, the functional Eq.(3.2) reads

$$l(\varphi) = \int_a^b dt k(x,y)\frac{d\varphi_t(x)}{dt}\frac{d\overline{\varphi}_t(y)}{dt}dxdy. \tag{3.16}$$

With the help of Eq.(3.15), we have:

$$l(\varphi) = \int_a^b dtk(x,y)\omega^2\psi(x)\overline{\psi}(y)dxdy. \tag{3.17}$$

Let us impose the normalization condition

$$\int \psi(x)\overline{\psi}(x)dx = 1. \tag{3.18}$$

The Euler-Lagrange equations for the functional Eq.(3.17) subject to the constraint Eq.(3.18) read

$$\int k(x,y)\psi(y)dy - \lambda\psi(x) = 0 \tag{3.19}$$

and the complex conjugate equation.

Notice that the energy eigenstates are normally sufficiently smooth functions. This in particular means that the kernel $k(x,y)$ can be chosen to be a singular generalized function. In this respect the situation is directly opposed to the considered case of macroscopic test-particles. In the latter case the state functions are singular and therefore the metric $k(x,y)$ must be a reasonably "good" function.

Assume then that the kernel $k(x,y)$ in Eq.(3.19) is given by

$$k(x,y) = -\Delta_x\delta(x-y) + V(x)\delta(x-y), \tag{3.20}$$

where Δ_x is the Laplacian and V is a given potential. In particular, the kernel Eq.(3.20) depends on the potential.

Notice that after integration by parts the metric defined by this kernel can be written in the following way:

$$\int k(x,y)\psi(x)\psi(y)dxdy = \int |\nabla\psi(x)|^2 dx + \int V(x)|\psi(x)|^2 dx. \tag{3.21}$$

Therefore, the Hilbert space with such an inner product is the (weighted) Sobolev space $H^1(R^3)$.

The equation Eq.(3.19) with the metric Eq.(3.20) reads

$$\int (-\Delta_x\delta(x-y) + (V(x)-\lambda)\delta(x-y))\psi(y)dy = 0. \tag{3.22}$$

"Integration by parts" applied twice gives

$$(-\Delta_x + V(x))\psi(x) = \lambda\psi(x), \tag{3.23}$$

i.e. the ordinary Schrödinger equation. As already mentioned, the physical interpretation of Eq.(3.23) with x as an abstract variable will be given in section 4.

Let us remark that no independent reason for the choice Eq.(3.20) of the metric was given. Therefore, the example only demonstrates the possibility of deducing the Schrödinger equation by variation of the functional $l(\varphi)$ under an appropriate choice of the metric.

Also, the normalization condition Eq.(3.18) seems to contradict the choice of the metric Eq.(3.20). It is important to realize, however, that the metric Eq.(3.20) is a *Riemannian* metric on $L_2(R^3)$. Such a metric is a bilinear functional on tangent spaces $H = T_\varphi L_2(R^3)$ to which $\frac{d\varphi(x,t)}{dt}$ belongs for each t and it should not be confused with the metric on $L_2(R^3)$ itself.

The examples demonstrate that, depending on the metric and the form of a state, the Euler-Lagrange equations for the functional $l(\varphi)$ yield both the classical motion of macroscopic particles along geodesics and the quantum mechanical behavior of microscopic particles according to the Schrödinger equation.

4 The emergence of spacetime: Discussion

Let us review the advocated scenario of emergence of the classical spacetime. We began with the observation that the Riemannian manifold structure of the classical (in general, curved) 3-space can be naturally recovered from the Hilbert space H of states of a macroscopic test-particle in the universe. In fact, the position of such a particle at any given moment of time is fixed. Therefore, H must contain spatially point-supported state functions, i.e. delta-functions $\delta_p(x)$ on the classical space. In the proposed formalism delta-functions and other generalized functions are on equal footing with the square-integrable functions. In particular, any such function in the formalism is an element of a Hilbert space. The correspondence $p \longrightarrow \delta_p(x)$ is one-to-one and allows us to identify the classical space with the submanifold $M_3 \subset H$ of all delta-functions in H. This identification is physically natural as the classical space is seen as such only when localized test-particles are used to observe it.

The above construction can be now reversed so that it becomes unnecessary to presuppose the existence of the classical space. The key observation is that the submanifold M_3 of H identified with the classical space is a set of *functions* rather than a set of *points* on which these functions are defined. In particular, the domain D of functions becomes irrelevant in defining the Riemannian manifold structure of M_3. Instead, such structure is defined by the embedding of M_3 into the abstract Hilbert space $\mathbf{S}$.

The embedding considered in the paper consists of a Hilbert realization $e_H : H \longrightarrow \mathbf{S}$ of $\mathbf{S}$ and of the identification map of M_3 with the submanifold of all delta-functions in H. Assume that the space H consists of generalized functions "of" three (or four) abstract variables, so that the domain D of the functions is, say, a ball in the Euclidean space R^3 (or R^4). Let us show that the developed formalism does not rely on the pre-existing classical space. We will prove that on the contrary:

(1) Various induced Riemannian metric on the "emerging" classical space M_3 can be derived by an appropriate choice of the space H of functions on D,

(2) The choice of H yields various topologically non-trivial spaces M_3, despite

the fact that the domain D of definition of functions in H is topologically trivial, and

(3) Both the Schrödinger equation for microscopic particles and the equation of geodesics for macroscopic test-particles can be derived without any appeal to the classical spacetime.

To prove (1) assume that the metric on H is defined by a kernel $k(x, y)$. By Eq.(2.18), the induced Riemannian metric on subspace M_3 is given by $g_{\alpha\beta} = \frac{\partial^2 k(x,y)}{\partial x^\alpha \partial y^\beta}$. Moreover, we verified in section 2 that *any* analytic Riemannian metric can be locally written is such a way provided $k(x, y)$ is appropriately chosen. It follows in particular that the metric on M_3 has nothing to do with the Euclidean metric on the domain $D \subset R^3$.

To prove (2) consider the following one-dimensional example. Let H be a Hilbert space of smooth functions φ on the interval $[0, 2\pi]$ such that

$$\varphi^{(n)}(0) = \varphi^{(n)}(2\pi) \tag{4.1}$$

for any φ in H and any order n of the derivative of φ. The derivatives in Eq.(4.1) are the usual one-sided derivatives. Consider the dual space H^* of functionals on H. Assume that the kernel of the metric on H^* is smooth and let us identify M with the submanifold in H^* consisting of the delta-functions. The space of parameters here is the interval $[0, 2\pi]$ which has a trivial topology. The condition Eq.(4.1) can be used to identify H with the space of smooth functions on the circle S^1. Respectively, H^* is the space of generalized functions "on" S^1. Let a be the angular parameter on S^1. Then the map $a \longrightarrow \delta(\theta - a)$ from S^1 into H^* is a parametrization of M which identifies M with the circle.

Notice once again that the non-trivial topology on M was obtained here by means of a condition Eq.(4.1) imposed on *functions* and despite the topological triviality of the space of parameters D. If no condition like Eq.(4.1) is imposed, the submanifold M becomes topologically trivial.

One can check in a similar way that topology of the submanifold $M_3 \subset H$ is not in general determined by the topology of the domain D of definition of functions in H. Instead, analogously to the Riemannian structure, the topology depends only on the way in which we identify M_3 as a submanifold of $\mathbf{S}$.

To verify (3) assume that a particular realization $e_H : H \longrightarrow \mathbf{S}$ is fixed and M_3 is identified with the submanifold of delta-functions in H. Let us construct a Riemannian 3-manifold N of *points* rather than functions (i.e. the one which is not a subset of H), which is isometric to M_3 and can be therefore identified with it. For this let us point out that the inverse of the parametrization map $P : a \longrightarrow \delta(x - a)$ identifies "pieces" of the manifold M_3 with the corresponding pieces of the space of parameters R^3 and induces the Riemannian manifold structure on the collection of the latter pieces. Notice that the space of parameters becomes then a *model space* for N but should not be confused with N itself. The derived manifold $N \cong M_3$ as well as its "parent" M_3 is now identified with the classical space. The parameters a^μ

become *coordinates* on the space, vectors tangent to M_3 become identified with the ordinary vectors and the induced metric becomes an (arbitrary) Riemannian metric on the classical space N (see section 2).

The way in which the manifold N has appeared is essential for understanding of why the results of section 3 do not rely on a pre-existing classical space. In particular, in formula Eq.(3.16) there is no need to assume that the variables x, y represent coordinates on the classical space. Under the given assumptions the Schrödinger equation Eq.(3.23) follows even if x is an unrelated to the classical space abstract variable with values in R^3. Of course, to extract a physical content from the obtained Schrödinger equation, one needs to be able to physically identify x. To measure x, a macroscopic measuring device, say, a macroscopic test-particle, must be used. The set of observed states of such a particle form the space M_3 of delta-functions parametrized by the abstract parameter a. As we just discussed, the parametrization map $a \longrightarrow \delta(x-a)$ induces the structure of a manifold N isometric to M_3. It becomes, therefore, possible to interpret a as coordinates on the classical space N. Respectively, assuming the abstract variable x in Eq.(3.23) stands for a, the standard interpretation of the Schrödinger equation follows. This, by the way, suggests that notation a rather than x should be used in Eq.(3.23) when ψ is considered to be a function on N.

Let us add a few comments to the above demonstration.

(I) The result (2) has an important implication: it shows that a change of the functional basis e_H on $\mathbf{S}$ may result in a change of topology/metric on the submanifold M_3. In fact, we saw that depending on the realization H the embedding of M_3 into H produces spaces of different Riemannian metric and topology. At the same time, all such infinite-dimensional separable Hilbert realizations H are known to be isomorphic. In particular, a change of realization is just a change of the functional basis e_H. In other words, what looks like a "coordinate transformation" on $\mathbf{S}$ (see Ref.[7]), can be observed as a change of topology on M.

(II) In (3), having introduced the manifold N, one can, if desirable, consider the set of delta-functions $\delta_p(x)$ on N. Clearly, there is a one-to-one correspondence between this set and M_3. Notice that the functions $\delta_p(x)$ have a different domain than delta-functions in H and should not be confused with the latter. Such a confusion would lead one to the idea that the classical space like N is still needed to define M_3.

(III) Let us also clarify the notion of an ϵ-neighborhood of a point on the space M_3. Such a notion is important in particular in defining what is *local* on M_3 without needing to refer to locality on spaces N or D. As M_3 is a Riemannian manifold, the ϵ-neighborhood of a point $\delta(x-a) \in M_3$ is naturally defined in terms of the Riemannian metric. We say in particular that two points $\delta(x-a)$, $\delta(x-b)$ in M_3 are ϵ-close if the distance between the points is less than ϵ. As the Riemannian metric on M_3 is induced by the embedding of M_3 into H, this also means that $||\delta(x-b)-\delta(x-a)||_H < \epsilon$. For any particular Hilbert space H the latter expression can be written in terms of the parameters. In particular, if the metric on H is defined by the kernel $e^{-\frac{1}{2}(x-y)^2}$, then $||\delta(x-b)-\delta(x-a)||_H < \epsilon$ means that $|2-2e^{-\frac{1}{2}(a-b)^2}| < \epsilon^2$ which is equivalent to $||a-b||_{R^3} < \epsilon_1$ for a uniquely defined ϵ_1.

The above discussion clarifies how the Riemannian geometry on M_3 can be introduced without any reference to the classical space. At the same time we see how a particular realization H of $\mathbf{S}$ has an "encoded" information about the classical space. Moreover, we see that the formalism allows one to mathematically derive both the classical and the quantum concepts (e.g., classical Riemannian space, state-functions defined on the classical space) from the concept of a Hilbert space of functions of abstract parameters. This derivation is *genuine* in the sense that it does not utilize the notion of a classical space.

We are therefore in the position to drop the assumption of a pre-existing space. Instead, the abstract infinite-dimensional separable Hilbert space $\mathbf{S}$ is taken to be a model of space adequate to the non-relativistic quantum theory. The classical space is then mathematically *derived* from $\mathbf{S}$ by the following steps:

(a) Select a realization $e_H : H \longrightarrow \mathbf{S}$ of $\mathbf{S}$ as a space of (generalized) functions "on", say, R^3,

(b) Define the submanifold M_3 of delta-functions in H with the induced Riemannian metric and identify it with the classical space.

As we already discussed, the same steps assure mathematical derivation of the classical spacetime. For this the space H must be a Hilbert space of generalized functions "of" 4-variables. However, the physical meaning of the elements of H in this case needs to be investigated.

Now that the mathematical derivation of the classical spacetime has been analyzed, let us turn to the physical aspects of the advocated model of emergence. The potentially useful in physics results obtained in the paper are as follows:

(A) Derivation of the Riemannian metric on the classical space from the metric on a Hilbert space of states,

(B) Derivation of the equation of geodesics on spacetime from the variational principle for the "square-length" functional restricted to paths on $M \subset H$, where M is the submanifold of delta-functions in H,

(C) Derivation of the Schrödinger equation by variation of the same functional restricted to the "stationary" paths taking values on the unit sphere in $L_2(R^3)$, and with tangent vectors in the weighted Sobolev space $H^1(R^3)$.

Each of these results poses several important questions that require further analysis. In particular, (A) relates the Riemannian metric on classical space with the Hilbert metric on a space of states. This relationship implies that different Riemannian metrics require different Hilbert spaces of states. How one would account then for the standard results in QM where the Hilbert space of states is always a space of square-integrable (with respect to some measure) functions?

To answer, recall that the Hilbert space with metric given by the kernel $g(x,y) = e^{-\frac{1}{2}(x-y)^2}$, where $(x-y)^2 = \delta_{\mu\nu}(x-y)^\mu(x-y)^\nu$, yields the standard Euclidean metric on the 3-space R^3 (see section 2). If one begins instead with the kernel $g_L(x,y) = \frac{L}{\sqrt{\pi}}e^{-L^2(x-y)^2}$, the induced metric differs from the Euclidean by a constant factor. In particular, it is conformally equivalent to the Euclidean metric. On the other hand, the sequence $\frac{L}{\sqrt{\pi}}e^{-L^2(x-y)^2}$ is a delta-convergent sequence as $L \longrightarrow \infty$. Let H_L be the Hilbert space with the metric given by the kernel g_L. Notice that for any L such a space contains δ-functions. At the same time, for a large enough L the L_2-norm and the H_L-norm of a square-integrable function on R^3 are practically indistinguishable. This can be used in conformally invariant theories to relate the metrics in the described way and, at the same time, to account for the standard predictions of QM.

There is, however, a more radical possibility. It is conceivable that the Riemannian metric on **S** is a tensor field, i.e. it changes as we move across **S**. In this case the metric in experiments producing improper and bound states could be different (see Ref.[9]).

As already discussed, the problem with derivation (B) is in the physical interpretation of functions of four variables needed to derive the equation of geodesics in spacetime (rather than just in classical space).

The derivation (C) is based on the following three assumptions:

(α) The Hilbert space **S** is realized as the space $L_2(R^3)$ of square-integrable functions on R^3.

(β) The admissible paths have the form $\varphi_t(x) = e^{-i\omega t}\psi(x)$, where ψ is unit-normalized. In other words, $\psi \in S^{L_2}$, where S^{L_2} is the unit-sphere in $L_2(R^3)$.

(γ) The Hilbert metric on spaces tangent to S^{L_2} is the Sobolev space metric defined by Eq.(3.21).

We remark here that the non-linearity of S^{L_2} is analogous to the non-linearity of the space M_3 of delta-functions in the Hilbert space of states of a macroscopic test-particle. The metric induced on M_3 was just the ordinary Riemannian metric on the classical 3-space. Here, for the dynamical equation to be correct, the similarly induced metric on S^{L_2} must be the Sobolev metric Eq.(3.21). The above "emerging" relationship between the classical, purely local dynamics of macroscopic particles and essentially non-local dynamics of microscopic particles warrants a thorough clarification and investigation.

What should be the next step in building a physically sound theory of emergence in accordance with the proposed scenario? Assume that indeed the abstract Hilbert space **S** or some generalization of it is an appropriate arena for all physical processes. Our primary goal must be then to derive the classical space and, more generally, the space and time by means of a physical process of emergence rather than just by a mathematical transformation. Such a process is likely to be related to the phenomena of decoherence and collapse that ensure spatial localization of state of macroscopic particles in the universe as described in introduction. However, for the emergence process to be truly independent of the emerging classical space, it must be formulated as a dynamical process on **S** itself.

No such dynamical mechanism for the advocated emergence scenario is offered in the paper. In particular, no dynamical reason for a particular choice of realization H of $\mathbf{S}$ is proposed. The model does indeed provide us with a way of deriving the Riemannian manifold structure on classical spacetime, but it does so "kinematically" rather than dynamically. Namely, the classical space (or spacetime) is derived in the paper through a choice of the functional basis e_H rather than as a result of a dynamical process.

The purpose of the model is to promote the quantum-mechanical Hilbert space of states to the status of a new, infinite-dimensional arena for modern physics. The developed formalism shows that the old fashioned classical space (or even spacetime) can be, in a sense, a "decohered" version of its newer counterpart. It remains to be seen, however, whether the formalism is adequate for the dynamical treatment of the problem of emergence and whether the above purpose can be fulfilled.

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