Coordinate Formalism in Abstract Hilbert Space: Kinematics of a Quantum Measurement

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Coordinate form of tensor algebra on an abstract (infinite-dimensional) Hilbert space is presented. The developed formalism permits one to naturally include the improper states in the apparatus of quantum theory. In the formalism the observables are represented by the self-adjoint extensions of Hermitian operators. The unitary operators become linear isometries. The unitary evolution and the non-unitary collapse processes are interpreted as isometric functional transformations. Several experiments are analyzed in the new context.

1 Introduction

The improper (i.e. non-square integrable) states play the essential role in quantum theory (QT): they serve the building blocks of the theory and simultaneously provide a bridge between the quantum and the classical worlds. In particular, a possibility (at least in principle) to measure precise positions of particles permits one to define the state function in the non-relativistic quantum mechanics (QM). A possibility to measure precise momenta of free particles is essential to define the scattering amplitude in quantum field theory (QFT). Simultaneously, such measurements create improper states thereby furnishing quantum particles with the classical-mechanical properties and providing the foundation of the familiar classical world.

The mathematical apparatus for working with the improper states seems to be well developed, yet somewhat unsatisfactory. This becomes especially clear in the QFT. In fact, the improper states are described by continuous linear functionals (generalized functions) defined on an appropriate space of functions. The passage to QFT involves working with the products of these functionals - the operation which up until now is not defined satisfactory, leading to divergences in QFT.

The famous measurement problem in QM also seems to be related to the lack of the appropriate treatment of generalized functions. In fact, whenever the spectrum of an observable is continuous, the complete system of orthogonal projectors needed to represent mathematically the von-Neumann reduction is not available. This fact

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together with the success of the rigged Hilbert space approach to the spectral analysis of operators (see [3], [4]) makes one wonder if the mathematics and, possibly, the physics of von Neumann reduction must be re-examined.

In the paper it is shown that the generalized functions and the square-integrable functions can be included on equal footing into the formalism of QT. This is done by developing a functional coordinate formalism on an abstract infinite-dimensional Hilbert space. In this we follow the earlier work [7],[8], however, the material is presented in a self-contained way.

In the formalism the usual QM observables are extended to self-adjoint operators acting on Hilbert spaces of generalized functions. The unitary evolution of quantum states is replaced with the more general *isometric* evolution. That is, the evolution operator while preserving the norm of a state is represented in general by a map between *different* functional Hilbert spaces.

The formalism demonstrates that the improper states naturally appear in QT if one is ready to admit that quantum-mechanical evolution can alter the Hilbert metric on the space of states. Once this is accepted, it becomes possible to treat the improper states and the square-integrable states equally and the invariant (functional tensor) approach to QM emerges. Ultimately the formalisms insists that QT must be expressed in the form independent of a choice of functional coordinates, i.e. in the functional tensor form.

Here is the plan of the paper. In section 2 after a brief discussion of the improper states in QM we introduce the coordinate formalism on abstract Hilbert space. The formalism is in essence a tensor algebra in the form that seems to be most appropriate to the case of infinite number of dimensions.

Several examples of Hilbert spaces containing singular generalized functions are given in section 3. The functional tensor form of a generalized eigenvalue problem for a linear operator is discussed in section 4. In section 5 the string basis of eigenvectors of a linear operator acting on the abstract Hilbert space is introduced. Such a basis is shown to generalize the usual basis of eigenvectors of a linear operator.

The second half of the paper deals with applications of the newly developed language in QM. The evolution of improper states in an experiment with a magnetic spectrometer is considered in section 6. The space of highly singular improper states that may be relevant to QT is derived in section 8. This is done by analyzing a thought experiment with a heavy harmonic oscillator. The measurement of electron's position in the two-slit experiment is analyzed in section 9. Here it is shown that the coordinate formalism may be useful in interpreting collapse of a state.

The results and open questions are briefly summarized in the conclusion.

2 Coordinate formalism on abstract Hilbert space

The eigenstates of position and momentum operators in QM are the most familiar examples of non-square integrable states. More generally the improper states ap-

pear as eigenstates of operators with a continuous spectrum. Such states cannot be directly treated within the standard formalism of Hilbert spaces L_2 of square-integrable functions. In particular, as the L_2-norm of improper states does not exist, one cannot rigorously speak of a unitary evolution of these states.

To remedy the situation one usually replaces the improper states with the appropriate states of square-integrable functions. In particular, one can put a particle in a box or describe a localized particle by the Gaussian function rather than by the δ-function. In doing so it is often assumed that the δ-function and the free particle state functions are only idealizations of the "real" state functions which are square-integrable. It is important to understand, however, that the entire quantum theory is based on the notion of such functions, similar to the fact that the classical particle mechanics is based on the notion of a geometric point. It seems therefore natural to deal with such functions directly without necessity to refute to the square-integrable functions.

The rigged Hilbert space approach to QM [3], [4] offers a possible way of dealing with the situation. However, the rigged Hilbert space (RHS) structure lacks the symmetry between the square-integrable states and the improper states. This follows already from the fact that the improper states in RHS have no norm. An alternative approach will be developed in this paper.

Let us first of all formulate clearly the guiding principles of the developing formalism. To begin with, we want to include in the formalism a variety of functional spaces rather than just the spaces L_2 of square-integrable functions. In particular, we want to consider spaces containing the eigenstates of position and momentum operators.

The symmetry between the square-integrable and the improper states will be the main principle of the formalism. It is therefore essential to include in the formalism various Hilbert spaces of functions including spaces of singular generalized functions on equal footing.

At this point we will assume that all such spaces are infinite-dimensional separable Hilbert spaces. In what follows such spaces will be called *separable Hilbert* or simply *Hilbert* spaces. As any two separable Hilbert spaces are isomorphic, it follows that the difference between them will be in the choice of Hilbert metric and elements.

It is also important to include functional transformations relating these spaces. We will assume that these transformations are isomorphisms of Hilbert spaces.

The isomorphisms of Hilbert spaces of functions shall be considered to be functional analogues of coordinate transformations on the Euclidean space R^n. That is, a functional version of the tensor algebra in coordinates must be developed.

The coordinate transformations are not going to be *unitary* in general as they will relate different Hilbert spaces of functions. Instead, such transformations will be *isometries*, that is, they will still preserve the norms of the elements while relating different Hilbert spaces of functions.

Let us now develop the details of such a formalism. What follows in this section and section 4 is an abbreviated version of [7] and [8]. The most important definitions and results of these papers are presented in a self-contained form.

As just discussed, we want to include in the coordinate formalism the arbitrary Hilbert spaces of functions. As all separable Hilbert spaces are isomorphic, it is possible to think that there is a single abstract Hilbert space **S**. Different Hilbert spaces of functions will be then *realizations* of **S**. We accept the following definitions:

Definition. A *string space* **S** is an abstract infinite-dimensional linear topological space isomorphic (that is, topologically linearly isomorphic) to a separable Hilbert space.

Definition. The elements of **S** are called strings and will be denoted by the capital Greek letters $\Phi, \Psi, ...$.

Let us remark here that despite their name, the string space and its elements at this stage have very little to do with the string theory.

Definition. A *Hilbert space of functions* is either a Hilbert space H, elements of which are equivalence classes of maps between two given subsets of R^n or the Hilbert space H^* dual to H. Two maps f, g are called *equivalent* if the norm of $f - g$ in H is zero.

Definition. Any Hilbert space of functions H is called a *coordinate space*. Elements of H are called *coordinates* (or *functional coordinates*) of strings and will be denoted by the small Greek letters $\varphi, \psi, ...$.

To identify strings with their coordinates we need a linear map $e_H : H \longrightarrow \mathbf{S}$ from a Hilbert space of functions H into the string space **S**. The action of e_H on $\varphi \in H$ will be written in one of the following ways:

$$e_H(\varphi) = (e_H, \varphi) = \int e_H(k)\varphi(k)dk = e_{Hk}\varphi^k. \tag{2.1}$$

The integral sign here is used as a notation for the action of e_H on elements of H and in general does not refer to an actual integration over k. We also use an obvious and convenient generalization of the Einstein's summation convention over the repeated indices k one of which is above and one below. Once again, only in special cases does this notation refer to an actual summation or integration over k.

Definition. A linear isomorphism e_H from a Hilbert space H of functions onto **S** is called a *string basis* on **S**. The inverse map $e_H^{-1} : \mathbf{S} \longrightarrow H$ is called a *linear coordinate system on* **S** (or a *linear functional coordinate system*).

By definition any string Φ is the image of a unique element $a \in H$, i.e.

$$\Phi = e_H(a) \tag{2.2}$$

for a unique $a \in H$. Also, if

$$e_H(a) = 0, \tag{2.3}$$

then $a = 0$.

Assume for example that $H = l_2$, where l_2 is the Hilbert space of square-summable sequences. Then the action of e_H on H reduces to the matrix multiplication of the row $(e_1, e_2, ...)$ of linearly independent vectors in $\mathbf{S}$ by the column of components of $\varphi \in l_2$. In this case the string basis can be identified with the basis in the ordinary sense.

It is worth noticing that the basis e_H defines the space H itself. In fact, it acquires the meaning only as a map on H.

Let $\mathbf{S}^*$ be the dual string space. That is, $\mathbf{S}^*$ is the space of all linear continuous functionals on strings.

Definition. A linear isomorphism of H^* onto $\mathbf{S}^*$ is called a *string basis on* $\mathbf{S}^*$.

We will denote such a basis by e_{H^*}. The "decomposition" of an element $F \in \mathbf{S}^*$ with respect to the basis will be written in one of the following ways:

$$F = e_{H^*}(f) = (e_{H^*}, f) = \int e_{H^*}(k) f(k) dk = e^k_{H^*} f_k. \tag{2.4}$$

Definition. The basis e_{H^*} is *dual* to the basis e_H if for any string $\Phi = e_{Hk}\varphi^k$ and for any functional $F = e^k_{H^*} f_k$ the following is true:

$$F(\Phi) = f(\varphi). \tag{2.5}$$

In general case we have

$$F(\Phi) = e_{H^*} f(e_H \varphi) = e^*_H e_{H^*} f(\varphi), \tag{2.6}$$

where $e^*_H : \mathbf{S}^* \longrightarrow H^*$ is the adjoint of e_H. Therefore, e_{H^*} is the dual string basis if $e^*_H e_{H^*} : H^* \longrightarrow H^*$ is the identity operator. In this case we will also write

$${e_{Hl}}^* e^k_{H^*} = \delta^k_l. \tag{2.7}$$

In special cases δ^k_l is the usual Krœnecker symbol or the δ- function.

Notice also that the action of F on Φ in any bases e_H on $\mathbf{S}$ and e_{H^*} on $\mathbf{S}^*$ can be written in the following way:

$$F(\Phi) = e^k_{H^*} f_k e_{Hl} \varphi^l = G(f, \varphi) = g^k_l f_k \varphi^l, \tag{2.8}$$

where G is a non-degenerate bilinear functional on $H^* \times H$.

By definition the string space $\mathbf{S}$ is isomorphic to a separable Hilbert space. We can assume then that $\mathbf{S}$ itself is an abstract Hilbert space. Any linear isomorphism $\pi_H : \mathbf{S} \longrightarrow H$ of $\mathbf{S}$ into a Hilbert space of functions can be thought of as inducing the Hilbert structure on H. Accordingly, we will assume that the string bases $e_H = {\pi_H}^{-1}$ are isomorphisms of Hilbert spaces. That is, a Hilbert structure on any coordinate space H will be assumed to be induced by a choice of string basis.

Let us assume that $\mathbf{S}$ is a real Hilbert space (generalization to the case of a complex Hilbert space will be obvious). We have:

$$(\Phi,\Psi)_S = \mathbf{G}(\Phi,\Psi) = G(\varphi,\psi) = g_{kl}\varphi^k\psi^l, \tag{2.9}$$

where $\mathbf{G} : \mathbf{S}\times\mathbf{S} \longrightarrow R$ is a bilinear form defining the inner product on $\mathbf{S}$ and $G : H\times H \longrightarrow R$ is the induced bilinear form. The expression on the right is a convenient form of writing the action of G on $H\times H$.

Theorem. The choice of a coordinate Hilbert space determines the corresponding string basis up to a unitary transformation.

Proof. Let e_H and $\widetilde{e}_H$ be two string bases on $\mathbf{S}$ with the same coordinate space H. Then $(\Phi,\Psi)_S = G(\varphi,\psi) = \widetilde{G}(\widetilde{\varphi},\widetilde{\psi}) = \widetilde{G}(U\varphi,U\psi)$, where U is a map of H into itself. As $\widetilde{G} = G$ we have $G(\varphi,\psi) = G(U\varphi,U\psi)$, that is, U is a unitary transformation on H. Therefore $e_H = \widetilde{e}_H U$, i.e. the basis e_H is determined up to a unitary transformation on H.

Definition. A string basis e_H in $\mathbf{S}$ will be called *orthogonal* if for any $\Phi,\Psi\in\mathbf{S}$

$$(\Phi,\Psi)_S = f_\varphi(\psi), \tag{2.10}$$

where f_φ is a *regular* functional and $\Phi = e_H\varphi$, $\Psi = e_H\psi$ as before. That is,

$$(\Phi,\Psi)_S = f_\varphi(\psi) = \int \varphi(x)\psi(x)d\mu(x), \tag{2.11}$$

where $\int$ here denotes an actual integral over a μ-measurable set $D\in R^n$ which is the domain of definition of functions in H.

If the integral in (2.11) is the usual Lebesgue integral and/or a sum over a discrete index x, the corresponding coordinate space is called an L_2-space. In this case we will also say that the basis e_H is *orthonormal*.

If the integral is a more general Lebesgue-Stieltjes integral, the coordinate space defined by (2.11) will be called an *L_2-space with the weight μ*.

The bilinear form $\mathbf{G} : \mathbf{S}\times\mathbf{S} \longrightarrow R$ generates a linear isomorphism $\widehat{\mathbf{G}} : \mathbf{S} \longrightarrow \mathbf{S}^*$ by $\mathbf{G}(\Phi,\Psi) = (\widehat{\mathbf{G}}\Phi,\Psi)$. In any basis e_H we have

$$(\Phi,\Psi)_S = (\widehat{\mathbf{G}}e_H\varphi, e_H\psi) = e_H^*\widehat{\mathbf{G}}e_H\varphi(\psi) = \widehat{G}\varphi(\psi), \tag{2.12}$$

where $\widehat{G} = e_H^*\widehat{\mathbf{G}}e_H$ maps H onto H^*. If e_H is orthogonal, then $\widehat{G}\varphi = f_\varphi$. In the case of an orthonormal basis e_H we will also write

$$(\Phi,\Psi)_S = \delta_{kl}\varphi^k\psi^l. \tag{2.13}$$

In a special case δ_{kl} can be the Krœnecker symbol or Dirac δ-function $\delta(k-l)$.

The form of writing in (2.13) makes it especially clear why the basis in the definition above is called orthonormal. In fact, in this case the kernel of the metric functional is a generalization of the Krœnecker symbol, which represents the Euclidean metric in an orthonormal basis.

It follows from the definition that if e_H is orthogonal, then H is a space $L_2(D,\mu)$ of square-integrable functions on a μ-measurable set $D\in R^n$. In particular, not

every coordinate Hilbert space H can produce an orthogonal string basis e_H. Assume, for example, that H is a space of functions on R which contains the δ-function as a coordinate φ of a string $\Phi \in \mathbf{S}$ (examples of such spaces will be given below in section 3). Then $\int \delta(k-l)\varphi(k)\varphi(l)dkdl$ is not defined, that is, δ-function is not a coordinate function of a string in orthonormal basis.

On the other hand, Hilbert spaces l_2 and $L_2(R)$ are examples of coordinate spaces that admit an orthonormal string basis.

This result does not contradict the well known existence of an orthonormal basis in any separable Hilbert space. In fact, the meaning of a string basis is quite different from the meaning of an ordinary basis on a Hilbert space. Namely, a string basis on $\mathbf{S}$ permits us to represent an invariant with respect to functional transformations object (string) in terms of a function, which is an element of a Hilbert space of functions. A basis on the latter space in turn permits us to represent this function in terms of numbers, components of the function in the basis.

Notice that formula (2.13) suggests that if H possesses an orhonormal basis e_H, then the chart $(\mathbf{S}, e_H^{-1})$ is analogous to a rectangular Cartesian coordinate systems in Euclidean space. More general formula (2.9) suggests that other string bases produce analogues of oblique Cartesian coordinate systems in Euclidean space.

Definition. A linear coordinate transformation on $\mathbf{S}$ is an isomorphism $\omega : \widetilde{H} \longrightarrow H$ of Hilbert spaces which defines a new string basis $e_{\widetilde{H}} : \widetilde{H} \longrightarrow \mathbf{S}$ by $e_{\widetilde{H}} = e_H \circ \omega$.

Let φ be coordinate of a string Φ in the basis e_H and $\widetilde{\varphi}$ its coordinate in the basis $e_{\widetilde{H}}$. Then $\Phi = e_H\varphi = e_{\widetilde{H}}\widetilde{\varphi} = e_H\omega\widetilde{\varphi}$. That is, $\varphi = \omega\widetilde{\varphi}$ by uniqueness of the decomposition. This provides the transformation law of string coordinates.

Let now $\Phi, \Psi \in \mathbf{S}$ and let $\mathbf{A}$ be a linear operator on $\mathbf{S}$. Let $\Phi = e_H\varphi, \Psi = e_H\psi$ with $\varphi, \psi \in H$. The scalar product $(\Phi, \mathbf{A}\Psi)_S$ is independent of a basis and in a basis e_H reduces to

$$(\varphi, A\psi)_H = (\widehat{G}\varphi, A\psi), \tag{2.14}$$

where $\widehat{G} : H \longrightarrow H^*$ defines the metric on H.

If $\omega : \widetilde{H} \longrightarrow H$ is a linear coordinate transformation and $\varphi = \omega\widetilde{\varphi}, \psi = \omega\widetilde{\psi}$, then

$$(\widehat{G}\varphi, A\psi) = (\widehat{G}\omega\widetilde{\varphi}, A\widetilde{\psi}) = (\omega^*\widehat{G}\omega\widetilde{\varphi}, \omega^{-1}A\omega\widetilde{\psi}) = (\widehat{G}_{\widetilde{H}}\widetilde{\varphi}, A_{\widetilde{H}}\widetilde{\psi}). \tag{2.15}$$

Therefore we have the following transformation laws:

$$\varphi = \omega\widetilde{\varphi} \tag{2.16}$$

$$\psi = \omega\widetilde{\psi} \tag{2.17}$$

$$\widehat{G}_{\widetilde{H}} = \omega^*\widehat{G}\omega \tag{2.18}$$

$$A_{\widetilde{H}} = \omega^{-1}A\omega. \tag{2.19}$$

More generally, we can consider $\mathbf{S}$ as a Hilbert manifold (modelled on itself). A Hilbert manifold is a topological space with a differentiable atlas where the charts take values in a Hilbert space [5]. Let then (U_α, π_α) be an atlas on $\mathbf{S}$.

Definition. A collection of quadruples $(U_\alpha, \pi_\alpha, \omega_\alpha, H_\alpha)$, where each H_α is a Hilbert space of functions and ω_α is an isomorphism of $\mathbf{S}$ onto H_α is called a *functional atlas* on $\mathbf{S}$. A collection of all compatible functional atlaces on $\mathbf{S}$ is called a *coordinate structure* on $\mathbf{S}$.

Definition. Let (U_α, π_α) be a chart on $\mathbf{S}$. If $p \in U_\alpha$, then $\omega_\alpha \circ \pi_\alpha(p)$ is called the *coordinate* of p. The map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ is called a *coordinate system.* The isomorphisms $\omega_\beta \circ \pi_\beta \circ (\omega_\alpha \circ \pi_\alpha)^{-1} : \omega_\alpha \circ \pi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \omega_\beta \circ \pi_\beta(U_\alpha \cap U_\beta)$ are called *functional (or string) coordinate transformations.*

As $\mathbf{S}$ is a differentiable manifold one can also introduce the tangent bundle structure $\tau : T\mathbf{S} \longrightarrow \mathbf{S}$ and the bundle $\tau^r_s : T^r_s\mathbf{S} \longrightarrow \mathbf{S}$ of tensors of rank (r, s).

A coordinate structure on the string space permits one to obtain a functional description of any tensor. Namely, let $\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s)$ be an (r, s)-tensor on $\mathbf{S}$.

Definition. The coordinate map $\omega_\alpha \circ \pi_\alpha : U_\alpha \longrightarrow H_\alpha$ for each $p \in U_\alpha$ yields the linear map of tangent spaces $d\rho_\alpha : T_{\omega_\alpha \circ \pi_\alpha(p)} H_\alpha \longrightarrow T_p\mathbf{S}$, where $\rho_\alpha = \pi_\alpha^{-1} \circ \omega_\alpha^{-1}$. This map is called a *local coordinate string basis on* $\mathbf{S}$.

Let $e_{H_\alpha} = e_{H_\alpha}(p)$ be such a basis and $e_{H^*_\alpha} = e_{H^*_\alpha}(p)$ be the corresponding dual basis. Notice that for each p the map e_{H_α} is a string basis as defined earlier. Therefore, the local dual basis is defined for each p as before and is a function of p.

We now have $F_i = e_{H^*_\alpha} f_i$, and $\Phi_j = e_{H_\alpha} \varphi_j$ for any $F_i \in T^*_p\mathbf{S}$, $\Phi_j \in T_p\mathbf{S}$ and some $f_i \in H^*_\alpha, \varphi_j \in H_\alpha$. Therefore

$$\mathbf{G}_p(F_1, ..., F_r, \Phi_1, ..., \Phi_s) = G_p(f_1, ..., f_r, \varphi_1, ..., \varphi_s) \tag{2.20}$$

defining component functions of the (r, s)-tensor $\mathbf{G}_p$ in the local coordinate basis e_{H_α}.

3 Hilbert spaces of generalized functions

So, where are we now? First of all, the class of Hilbert spaces of square-integrable functions used in QT is replaced with a larger class of all Hilbert spaces of functions (as defined in the previous section).

All such spaces are isomorphic and this usually creates a wrong impression that a single Hilbert space is sufficient. However, no particular Hilbert space of functions is large enough to contain all functions useful in applications. In particular, the singular generalized functions should not be left aside.

Secondly, the coordinate formalism on an abstract Hilbert space is developed, which relates different spaces of functions (coordinate spaces). This formalism can be extended to an arbitrary abstract Hilbert manifold (see [7]). In a nutshell the coordinate formalism is nothing but the algebra of tensors on infinite-dimensional manifolds in a coordinate form.

In recent years the idea of describing mathematics in coordinates was not very popular and the "coordinate free" approach has been used. In case of manifolds of a

finite dimension this is well motivated as coordinates are "all alike" representing the points of a manifold by columns of numbers. However, functions representing the points of infinite-dimensional manifolds can have very different analytic properties. As a result, the coordinate approach becomes again important.

One of the main motivations for developing the presented formalism was to include singular generalized functions in the apparatus of QT. The physical consequences of this will be discussed in the following sections. Here we present several examples of Hilbert spaces of generalized functions.

Example 1. Consider first the space $H = \{\varphi : \varphi = \sum_{k=-\infty}^{\infty} c_k \delta(x-k), c_k \in R\}$ of linear combinations of δ-functions of a single variable with the inner product given by

$$(\varphi, \psi)_H = \int e^{-(x-y)^2-(y-z)^2} \varphi(x)\psi(z)dxdydz. \tag{3.21}$$

Integration with respect to x and z is understood here in the sense of section 2, i.e. as the action of a bilinear functional on $H \times H$.

As

$$\int e^{-(a-y)^2} e^{-(y-b)^2} dy = \sqrt{\frac{\pi}{2}} e^{-\frac{(a-b)^2}{2}}, \tag{3.22}$$

we have for the inner product of $\varphi = \sum_{k=-\infty}^{\infty} c_k \delta(x-k)$ and $\psi = \sum_{l=-\infty}^{\infty} d_l \delta(x-l)$:

$$(\varphi, \psi)_H = \sqrt{\frac{\pi}{2}} \sum_{k,l=-\infty}^{\infty} c_k d_l e^{-\frac{1}{2}(k-l)^2}. \tag{3.23}$$

Theorem. The bilinear functional (3.21) is symmetric and positive definite, i.e. it does indeed define the inner product on H.

Proof. The functional is obviously symmetric. Observe now that

$$(\varphi, \psi)_H = (\rho\varphi, \rho\psi)_{L_2}, \tag{3.24}$$

where

$$(\rho\varphi)(y) = \int e^{-(x-y)^2} \varphi(x)dx \tag{3.25}$$

is a linear map from H into $L_2(R)$. This map is injective as the functions in the set

$$\left\{ e^{-(x-k)^2} : k \in Z \right\}, \tag{3.26}$$

where Z is the set of integers, are linearly independent. The positive definiteness follows now from injectivity of ρ and (3.24).

Theorem. The space H with the inner product (3.21) is Hilbert.

Proof. We need to check that H is complete and separable. Assume therefore that

$$\varphi^N(x) = \sum_{k=-\infty}^{\infty} c_k^N \delta(x-k) \tag{3.27}$$

is a fundamental sequence, i.e. $\left\|\varphi^N - \varphi^M\right\|_H \longrightarrow 0$ as $N, M \longrightarrow \infty$. Here $\|\varphi\|_H^2 = (\varphi, \varphi)_H$.

By (3.23) we then have

$$\left\|\varphi^N - \varphi^M\right\|_H^2 = \sqrt{\frac{\pi}{2}} \sum_{k,m=-\infty}^{\infty} (c_k^N - c_k^M)(c_m^N - c_m^M) e^{-\frac{1}{2}(k-m)^2} \longrightarrow 0 \tag{3.28}$$

as $N, M \longrightarrow \infty$. It follows that $c_k^N - c_k^M \longrightarrow 0$ for any k as $N, M \longrightarrow \infty$.

By completeness of the set of real numbers R for any k there exists c_k such that $c_k^N \longrightarrow c_k$. Consider then $\varphi(x) = \sum_{k=-\infty}^{\infty} c_k \delta(x-k)$. The standard argument shows now that $\varphi \in H$ and $\left\|\varphi^N - \varphi\right\|_H \longrightarrow 0$. Finally, the set of elements of H with the rational coefficients c_k is dense in H, i.e. H is separable.

Notice in particular that the space H can be identified with the space of sequences c_k such that

$$\sum_{k=-\infty}^{\infty} c_k c_m e^{-\frac{1}{2}(k-m)^2} < \infty. \tag{3.29}$$

Example 2. Consider the space $H = \left\{\varphi : \varphi = \sum_{k=0}^{\infty} c_k \delta^{(k)}(x), c_k \in R\right\}$, where $\delta^{(k)}(x)$ denotes the derivative of order k of the δ-function of $x \in R$. For the metric on H let us choose the following bilinear functional:

$$(\varphi, \psi)_H = \int e^{-(x-y)^2} e^{y^2} e^{-(y-z)^2} \varphi(x)\psi(z) dx dy dz. \tag{3.30}$$

The factor e^{y^2} is chosen for simplicity of the final expression.

Theorem. The space H with the metric (3.30) is Hilbert.

Proof. For the elements $\varphi(x) = \sum_{k=0}^{\infty} c_k \delta^{(k)}(x)$, $\psi(x) = \sum_{m=0}^{\infty} d_m \delta^{(m)}(x)$, using definition (3.30) and "integration by parts" we have:

$$(\varphi, \psi)_H = \sum_{k,m=0}^{\infty} c_k d_m (-1)^k (-1)^m \times$$
$$\times \int \left(\frac{d^k}{dx^k} e^{-(x-y)^2}\right) e^{y^2} \left(\frac{d^m}{dz^m} e^{-(y-z)^2}\right) \delta(x)\delta(z) dx dy dz. \tag{3.31}$$

Also, from the definition of Hermite polynomials (see (7.95)) we have:

$$\frac{d^k}{dx^k} e^{-(x-y)^2}|_{x=0} = \frac{d^k}{d\xi^k} e^{-\xi^2}|_{\xi=-y} = (-1)^k H_k(\xi) e^{-\xi^2}|_{\xi=-y} = H_k(y) e^{-y^2}. \tag{3.32}$$

Similarly,

$$\frac{d^m}{dz^m} e^{-(y-z)^2}|_{z=0} = H_m(y) e^{-y^2}. \tag{3.33}$$

Let us use the orthogonal property of Hermite polynomials:

$$\int e^{-y^2} H_k(y) H_m(y) dy = N_{km} \delta_{km}, \tag{3.34}$$

where N_{km} are constants. Formulas (3.32), (3.33) and (3.34) then give

$$(\varphi, \psi)_H = \sum_{k=0}^{\infty} N_k^2 c_k d_k, \tag{3.35}$$

where $N_k = N_{kk}$. It follows that the space H can be identified with the space l_2 of square-summable sequences. In particular, H is a Hilbert space.

Notice also that only when the number of terms in $\sum_{k=0}^{\infty} c_k \delta^{(k)}(x)$ is finite does the sum represent a functional concentrated at $x = 0$ (see [2]).

More generally let us use the next example to show how one can practically construct large Hilbert spaces containing standard spaces of generalized functions as topological subspaces.

Example 3. Let W be the Schwartz space of infinitely differentiable rapidly decreasing functions on R. That is, functions $\varphi \in W$ satisfy inequalities of the form $\left| x^k \varphi^{(n)}(x) \right| \leq C_{kn}$ for some constants C_{kn} and any $k, n = 0, 1, 2, ...$. Let W^* be the dual space of continuous linear functionals on W. Let us find a Hilbert space which contains W^* as a subset. For this consider a linear transformation ρ from the space $L_2(R)$ into W given by

$$(\rho f)(x) = \int f(y) e^{-(x-y)^2 - x^2} dy \tag{3.36}$$

for any $f \in W^*$.

Theorem. $\rho(L_2(R)) \subset W$ and ρ is injective.

Proof. Clearly (3.36) defines an infinitely differentiable function decreasing faster than any power of $1/|x|$.

To check injectivity of ρ assume $\rho f = 0$. That is,

$$\int f(y) e^{-(x-y)^2} dy = 0. \tag{3.37}$$

Differentiating an arbitrary number k of times under the integral sign and changing to $z = x - y$ we have:

$$\int f(x - z) \frac{d^k}{dz^k} e^{-z^2} dz = 0. \tag{3.38}$$

Let us use the fact that

$$\frac{d^k}{dz^k} e^{-z^2} = H_k(z)(-1)^k e^{-z^2}, \tag{3.39}$$

where $H_k(z)$ are Hermite polynomials. It follows that

$$\int f(x-z)e^{-\frac{z^2}{2}}\varphi_k(z)dz = 0, \tag{3.40}$$

where $\varphi_k(z) = H_k(z)e^{-\frac{z^2}{2}}$ is a complete orthonormal system of functions in $L_2(R)$. That is, $f = 0$ in $L_2(R)$ proving injectivity of ρ.

Now, ρ induces a Hilbert metric on $H = \rho(L_2(R))$ by $(\varphi, \psi)_H = (\rho^{-1}\varphi, \rho^{-1}\psi)_{L_2}$. In particular, H with this metric is Hilbert.

Theorem. The embedding $H \subset W$ is continuous.

Proof. Topology on W may be defined by the countable system of norms

$$\|\varphi\|_p = \sup_{x\in R; k,q\leq p} \left|x^k\varphi^{(q)}(x)\right|, \tag{3.41}$$

where k, q, p are nonnegative integers and $\varphi^{(q)}$ is the derivative of φ of order q. For any $\varphi \in H$ by the Schwarz inequality we have:

$$\begin{aligned}\left|x^k\varphi^{(q)}(x)\right| &= \left|\int x^k f(y)\frac{d^q}{dx^q}e^{-(x-y)^2-x^2}dy\right| \\ \leq \left(\int x^{2k}P_q^2(x)e^{-2(x-y)^2-2x^2}dy\right)^{\frac{1}{2}} &\|f\|_{L_2} = M_{k,q}\|f\|_{L_2},\end{aligned} \tag{3.42}$$

where $\rho f = \varphi$, P_q is a polynomial of degree q and $M_{k,q}$ are constants depending only on k, q. This proves that topology on H is stronger then topology on W, i.e. $H \subset W$ is a continuous embedding.

It follows in particular that $W^* \subset H^*$ as a set. In fact, any functional continuous on W will be continuous on H.

Let us point out that the method used to obtain the Hilbert space H^* is quite general. It can be summarized as follows. First, find a linear injection from a standard Hilbert space (say, $L_2(R)$) into itself. This injection induces a Hilbert structure on the image H. Then construct the conjugate Hilbert space H^*. By choosing the injection to a small enough subspace H, one obtains as large H^* as one wishes.

The metric on H^* induced by $\rho^* : H^* \longrightarrow L_2^*$ in the example is given by

$$(f, g)_{H^*} = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}f(x)g(z)dydxdz. \tag{3.43}$$

Let us see in particular how the norm of singular functionals in H^* becomes finite. Consider for example the norm of the δ-function. We have:

$$(\delta, \delta)_{H^*} = (\rho\rho^*\delta, \delta) = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}\delta(x)\delta(z)dydxdz = \frac{\sqrt{\pi}}{2}. \tag{3.44}$$

The fact that the norm of the δ-function in H^* is finite is of course related to the fact that the metric $g_{xz} = \int e^{-(x-y)^2-x^2} e^{-(y-z)^2-z^2} dy$ is a smooth function of x and z and is capable of "compensating" singularities of the product of two δ-functions. Conversely, in case of the coordinate space $L_2(R)$ the metric g_{xz} is equal to $\delta(x-z)$. Therefore the norm of the δ-function is not defined. That is, δ-function can not be the coordinate of a string in this case.

More generally, it is easy to see that by "smoothing" the metric we extend the class of (generalized) functions for which the norm defined by this metric is finite. The same is true if we "improve" the behavior of the metric for large $|x|$. Conversely, by "spoiling the metric" we make the corresponding Hilbert space "poor" in terms of a variety of the elements of the space.

This shows once again that the developed coordinate formalism deals with a different "quality" of elements than its finite dimensional counterpart. By changing a string basis we change such properties of coordinates of a string as smoothness, convergence, etc. These properties make no sense for the finite dimensional coordinates of a point which simply are columns of numbers.

4 Generalized eigenvalue problem

Let A be a linear operator on a linear topological space V. If V is a finite-dimensional unitary space and A is, say, Hermitian, then a basis in V exists, such that each vector of it is an eigenvector of A. If V is infinite-dimensional, this statement is no longer true. Yet quite often there exists a complete system of "generalized eigenfunctions" of A in the sense of the definition below (see [3]):

Definition. A linear functional F on V, such that

$$F(A\Phi) = \lambda F(\Phi) \tag{4.45}$$

for every $\Phi \in V$, is called a *generalized eigenfunction of A corresponding to the eigenvalue λ.*

It is usually assumed that V in (4.45) is a space of test functions, i.e. a linear topological space of infinitely differentiable (and "good behaving" at infinity) functions used in the theory of generalized functions.

Notice however, that the definition above makes sense in a more general setting when F and Φ are elements of a pair of dual spaces. Assume then that V is the string space $\mathbf{S}$ and e_H is a string basis on $\mathbf{S}$. Assume F is a generalized eigenfunction of a linear operator $\mathbf{A}$ on $\mathbf{S}$. Then

$$F(\mathbf{A}\Phi) = \lambda F(\Phi), \tag{4.46}$$

that is,

$$e_H^* F(e_H^{-1} \mathbf{A} e_H \varphi) = \lambda e_H^* F(\varphi), \tag{4.47}$$

where $e_H\varphi = \Phi$ and $e_H^{-1}\mathbf{A}e_H$ is the representation of $\mathbf{A}$ in the basis e_H.

By defining $e_H^* F = f$ and $A = e_H^{-1}\mathbf{A}e_H$ we have

$$f(A\varphi) = \lambda f(\varphi). \tag{4.48}$$

Notice that the last equation describes not just one eigenvalue problem, but a family of such problems, one for each string basis e_H. As we change e_H, the operator A in general changes as well, as do the eigenfunctions f.

Example. Let H be the Hilbert space of functions dual to the space H^* of functionals $f(x) = \sum_{p=-\infty}^{\infty} c_p e^{ipx}$, $c_p \in R$ with the metric to be defined later (see (6.92)). Consider the operator of differentiation $A = i\frac{d}{dx}$ on H. The generalized eigenvalue problem for A is

$$f(i\frac{d}{dx}\varphi) = pf(\varphi). \tag{4.49}$$

The equation (4.49) must be satisfied for every φ in H. The functionals

$$f(x) = e^{-ipx} \tag{4.50}$$

are the eigenvectors of A. Let us now consider a coordinate change $\rho : H \longrightarrow \widetilde{H}$ given by the Fourier transform. Here $\widetilde{H}$ is dual to the Hilbert space of functionals $g(x) = \sum_{k=-\infty}^{\infty} c_k \delta(x-k)$ considered in example 1 of section 3. We have:

$$\psi(k) = (\rho\varphi)(k) = \int \varphi(x) e^{ikx} dx. \tag{4.51}$$

The Fourier transform induces a Hilbert structure on H. Relative to this structure ρ is an isomorphism of the Hilbert spaces $\widetilde{H}$ and H. The inverse transform is given by

$$(\omega\psi)(x) = \frac{1}{2\pi}\int \psi(k) e^{-ikx} dk. \tag{4.52}$$

According to (4.47) the eigenvalue problem in new coordinates is

$$\omega^* f(\rho A \omega \psi) = p\omega^* f(\psi). \tag{4.53}$$

We have:

$$A\omega\psi = i\frac{d}{dx}\frac{1}{2\pi}\int \psi(k) e^{-ikx} dk = \frac{1}{2\pi}\int k\psi(k) e^{-ikx} dk. \tag{4.54}$$

Therefore,

$$(\rho A \omega \psi)(k) = k\psi(k). \tag{4.55}$$

So, the eigenvalue problem in new coordinates is as follows:

$$g(k\psi) = pg(\psi). \tag{4.56}$$

Thus, we have the eigenvalue problem for the operator of multiplication by the variable. The eigenfunctions here are given by

$$g(k) = \delta(k - p). \tag{4.57}$$

Notice that $g = \omega^* f$ is as it should be. Indeed,

$$(\omega^* f)(k) = \frac{1}{2\pi} \int f(x) e^{ikx} dx = \frac{1}{2\pi} \int e^{-ipx} e^{ikx} dx = \delta(k - p). \tag{4.58}$$

As a result, the eigenvalue problems (4.49), and (4.56) can be considered as two coordinate expressions of a single functional tensor eigenvalue problem

$$F(\mathbf{A}\Phi) = \lambda F(\Phi) \tag{4.59}$$

for an operator $\mathbf{A}$ on $\mathbf{S}$.

5 The string basis of eigenvectors of a linear operator

Having introduced generalized eigenvectors it is natural to ask whether we can make a basis out of them. In what follows we assume that $\mathbf{A}$ is a bounded linear operator on $\mathbf{S}$.

Definition 1. A string basis e_H is the *proper basis* of a linear operator $\mathbf{A}$ on $\mathbf{S}$ with eigenvalues (eigenvalue function) $\lambda = \lambda(k)$, if

$$\mathbf{A} e_H(\varphi) = e_H(\lambda\varphi) \tag{5.60}$$

for any $\varphi \in H$.

As any string basis, the proper basis of $\mathbf{A}$ is a linear map from H onto $\mathbf{S}$ and a numeric function λ is defined on the same set as functions $\varphi \in H$.

By rewriting (5.60) as

$${e_H}^{-1} \mathbf{A} e_H(\varphi) = \lambda\varphi \tag{5.61}$$

we see that the problem of finding a proper basis of $\mathbf{A}$ is equivalent to the problem of finding such a string basis e_H in which the action of $\mathbf{A}$ reduces to multiplication by a function λ. In the particular case of an l_2-basis this yields the classical problem of finding a basis of eigenvectors of a linear operator.

The operator of multiplication by a function does not always map a Hilbert space H into itself. It is then useful to generalize the definition above.

Notice, that for any $F \in \mathbf{S}^*, \Phi \in \mathbf{S}$ and $\mathbf{A} : \mathbf{S} \longrightarrow \mathbf{S}$ we have

$$F(\mathbf{A}\Phi) = F(\mathbf{A} e_H \varphi) = F(e_{\widetilde{H}} e_{\widetilde{H}}^{-1} \mathbf{A} e_H \varphi) = e_{\widetilde{H}}^* F(e_{\widetilde{H}}^{-1} \mathbf{A} e_H \varphi) = f(A\varphi), \tag{5.62}$$

where $f = e^*_{\widetilde{H}} F \in \widetilde{H}^*$ and $A : H \longrightarrow \widetilde{H}$ is given by $A = e^{-1}_{\widetilde{H}} \mathbf{A} e_H$.

We then have the following

Definition 2. The representation $A = e^{-1}_{\widetilde{H}} \mathbf{A} e_H$ of $\mathbf{A}$ is called *proper* if for any $F \in \mathbf{S}^*$, $\Phi \in \mathbf{S}$ we have

$$F(\mathbf{A}\Phi) = f(A\varphi) = f(\lambda\varphi), \tag{5.63}$$

where $f = e^*_{\widetilde{H}} F$, $\varphi = e^{-1}_H \Phi$ and λ is a function such that the operator of multiplication by λ maps H into $\widetilde{H}$.

If $H = \widetilde{H}$ this is equivalent to the definition 1 above.

Let us compare the notions of a proper basis and of the ordinary basis of eigenvectors of an Hermitian operator $\mathbf{A}$ in more detail. For this assume first that $H = l_2$. Let e_{l_2} be a proper basis of $\mathbf{A}$ existence of which is assumed. Let $\varphi = (\varphi_1, \varphi_2, ...) \in l_2$. The action of e_{l_2} on φ reduces to the matrix multiplication of the matrix $e_{l_2} = (e_1, e_2, ...)$ of linearly independent vectors by the column of components of φ. That is, $e_{l_2}\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$. Notice that convergence of the series is assured by the continuity of e_{l_2}. We have:

$$\mathbf{A}e_{l_2}\varphi = e_{l_2}(\lambda\varphi) = e_{l_2}(\lambda_1\varphi_1, \lambda_2\varphi_2, ...) = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \quad . \tag{5.64}$$

Assume now that the ordinary basis $e_{l_2} = (e_1, e_2, ...)$ of eigenvectors of $\mathbf{A}$ exists. Then $\varphi = e_1\varphi_1 + e_2\varphi_2 + ...$ and

$$\mathbf{A}(e_1\varphi_1 + e_2\varphi_2 + ...) = \lambda_1 e_1\varphi_1 + \lambda_2 e_2\varphi_2 + ... = e_1\lambda_1\varphi_1 + e_2\lambda_2\varphi_2 + ... \quad . \tag{5.65}$$

Comparison of (5.64) and (5.65) shows that when $H = l_2$, the proper basis of $\mathbf{A}$ can be identified with the basis of eigenvectors of $\mathbf{A}$ in the ordinary sense.

In the proper basis with $H = l_2$ we have:

$$F(\mathbf{A}\Phi) = f(\lambda\varphi) = (\psi, \lambda\varphi)_{l_2} = \sum_{i=1}^{\infty} \lambda_i \psi_i \varphi_i, \tag{5.66}$$

where $f = (\psi, \cdot)_{l_2}$. In particular, ψ in (5.66) can be chosen to be an eigenvector of the operator of multiplication by the function λ.

Assume now that A is an Hermitian operator in $L_2(R)$. Then

$$(f, Ag)_{L_2} = (Af, g)_{L_2} \tag{5.67}$$

for any f and g in the domain of A. Therefore,

$$(\Phi, \mathbf{A}\Psi)_S = (\mathbf{A}\Phi, \Psi)_S, \tag{5.68}$$

where $\Phi = e_{L_2} f, \Psi = e_{L_2} g, \mathbf{A} = e_{L_2} A {e_{L_2}}^{-1}$, and e_{L_2} is an L_2-string basis on $\mathbf{S}$.

It follows that the same will be true in any string basis e_H, that is,

$$(\varphi, A_H\psi)_H = (A_H\varphi, \psi)_H, \tag{5.69}$$

where $\Phi = e_H\varphi, \Psi = e_H\psi$, and $\mathbf{A} = e_H A_H {e_H}^{-1}$.

Hermicity of $\mathbf{A}$ leads to a severe restriction on possible spaces H on which the proper basis of $\mathbf{A}$ is defined. In fact, if e_H is a proper basis of an Hermitian operator $\mathbf{A}$ on $\mathbf{S}$, then

$$(\Phi, \mathbf{A}\Psi)_S = (e_H\varphi, \mathbf{A}e_H\psi)_S = (\varphi, \lambda\psi)_H = \int g(x,y)\varphi(x)\lambda(y)\psi(y)dxdy. \quad (5.70)$$

In agreement with section 4, the integral symbol is used here for the action of the bilinear metric functional G with the kernel g. Hermicity gives then

$$\int g(x,y)(\lambda(x) - \lambda(y))\varphi(x)\psi(y)dxdy = 0 \quad (5.71)$$

for any $\varphi, \psi \in H$. It follows that the bilinear functional defined by $g(x,y)(\lambda(x) - \lambda(y))$ is equal to zero.

Assume first that H is a Hilbert space of sequences and no two values of λ are equal. Then $g(x,y) = a_x\delta_{xy}$, where δ_{xy} is the Krœnecker δ-symbol. It follows that the coordinate space H must be the space l_2 with the weight a_x.

Recall that when H is a space of sequences, the proper basis of an operator $\mathbf{A}$ on $\mathbf{S}$ can be identified with the ordinary basis of eigenvectors of $\mathbf{A}$. The obtained result is then equivalent to the well known orthogonality of the eigenvectors corresponding to different eigenvalues of the Hermitian operator $\mathbf{A}$.

A similar situation arises when H is a Hilbert space of functions on R. Assume for simplicity that $\lambda(x) = x$ and that H contains as a dense subset the space K of infinitely differentiable functions of bounded support on R. Then one can show [9] that $g(x,y) = a(x)\delta(x-y)$, i.e. H must coincide with the space $L_2(R)$ with the weight function $a(x)$.

We conclude that the proper bases are orthogonal in the considered cases (as defined in section 2). The result can be generalized to the case of functions of several variables on a domain in $D \subset R^n$.

This conclusion seems to favor the use of coordinate spaces L_2 when working with Hermitian operators. Notice however, that in the proper basis of $\mathbf{A}$ with $H = L_2(R)$ we have

$$(\Phi, \mathbf{A}\Psi)_S = (f, \lambda g)_{L_2(R)} = \int f(x)\lambda(x)g(x)dx, \quad (5.72)$$

where $f, g \in L_2(R)$. Therefore, it is impossible for $f(x)$ to be a (generalized) eigenvector of the operator A of multiplication by the function $\lambda(x)$. Indeed, the eigenvectors of such an operator would be δ-functions and the latter ones do not belong to $L_2(R)$.

From section 4 we recall that the eigenvectors of $A = \lambda\cdot$ acting on $L_2(R)$ can be defined as functionals F on a space V of functions, such that

$$F(A\varphi) = \lambda_0 F(\varphi) \quad (5.73)$$

for any $\varphi \in V$ and some $\lambda_0 \in R$.

To include such functionals in the formalism we need to consider Hilbert spaces containing "more" functions than $L_2(R)$. By the above this in general requires consideration of non-Hermitian operators.

Example. We know that $A_H = x$ (operator of multiplication by the variable) has no eigenvectors on $L_2(R)$. Consider then a different Hilbert space H of functions of a real variable with a metric G. We have:

$$(G\varphi, A_H\psi) = \int g(k,m)\varphi(k)m\psi(m)dkdm. \tag{5.74}$$

By (5.71)

$$\int g(k,m)\varphi(k)m\psi(m)dkdm \neq \int g(k,m)k\varphi(k)\psi(m)dkdm, \tag{5.75}$$

that is, A_H is not Hermitian unless H is an L_2-space. On the other hand, if $g(k,m)$ is an ordinary function which defines a regular functional, then obviously for any functions φ, ψ

$$\int g(k,m)\varphi(k)m\psi(m)dkdm = \int mg(k,m)\varphi(k)\psi(m)dkdm. \tag{5.76}$$

So by taking

$$F = \int g(k,m)\varphi(k)dk, \tag{5.77}$$

we have:

$$F(m\psi) = (mF)(\psi). \tag{5.78}$$

If the functional $F = \int g(k,m)\varphi(k)dk$ above is not regular, we can define the action of A_H on it by taking

$$(mF)(\varphi) = F(m\varphi). \tag{5.79}$$

The resulting operator of multiplication by m is defined on H^* and is self-adjoint in the sense that (5.79) is satisfied for all possible φ, F.

To avoid confusion let us accept the following definitions.

Definition. Let A be a continuous linear operator which maps a space H into a space $\widetilde{H}$. Then the *adjoint* A^* of operator A maps the space $\widetilde{H}^*$ into the space H^* according to

$$(A^*f, \varphi) = (f, A\varphi) \tag{5.80}$$

for any $\varphi \in H, f \in \widetilde{H}^*$. In particular, if the extension of A to H^* (under identification $L_2 = L_2^*$) exists and $(Af, \varphi) = (f, A\varphi)$ for any $\varphi \in H, f \in H^*$, the operator A will be called self-adjoint.

Definition. Let A be a continuous linear operator on a Hilbert space H. Then the *Hermitian conjugate* operator A^+ of A is defined on H by

$$(A^+\varphi, \psi)_H = (\varphi, A\psi)_H, \tag{5.81}$$

for any $\varphi, \psi \in H$. In particular, if $A^+ = A$ on H, then A is called Hermitian.

Let now A be given on H and let $\widehat{G} : H \longrightarrow H^*$ define a metric on H. Then

$$(\widehat{G}\varphi, A\psi) = (A^*\widehat{G}\varphi, \psi) = (\widehat{G}A^+\varphi, \psi) \tag{5.82}$$

and the relationship between the operators is as follows:

$$A^+ = \widehat{G}^{-1}A^*\widehat{G}. \tag{5.83}$$

In particular, self-adjoint operators defined on L_2 spaces can be identified with Hermitian operators.

Let us now return to the example under investigation. To be specific, assume that H is the Hilbert subspace $H \subset L_2(R)$ of functions as in example 3 of section 3. The space H consists of infinitely differentiable functions only. The space H^* contains singular distributions. If $\widehat{G}$ denotes the metric on H, then the inner product on H^* is defined by

$$(f, g)_{H^*} = (\widehat{G}^{-1}f, g) = \int e^{-(x-y)^2-x^2}e^{-(y-z)^2-z^2}f(x)g(z)dydxdz. \tag{5.84}$$

Unlike the case of $L_2(R)$-space it is possible to find such a function $\varphi \in H$ that $\widehat{G}\varphi \in H^*$ in the bilinear expression

$$(\widehat{G}\varphi, x\psi) \tag{5.85}$$

is a (generalized) eigenvector of x. We can take, for example, $(\widehat{G}\varphi)(x) = \delta(x - \lambda_0)$. With such a choice we have:

$$(\widehat{G}\varphi, x\psi) = \lambda_0(\widehat{G}\varphi, \psi) = \lambda_0(f, g)_{L_2}, \tag{5.86}$$

where as before f and g are L_2-components of the strings Φ, Ψ and φ, ψ are the corresponding components in a basis e_H. By rewriting the original bilinear expression in $L_2(R)$-basis we also have:

$$(\widehat{G}\varphi, x\psi) = (f, A_{L_2}g)_{L_2}. \tag{5.87}$$

Here A_{L_2} is the representation in $L_2(R)$-basis of the operator $\mathbf{A}$ acting on $\mathbf{S}$, whose representation in the e_H-basis is x.

Therefore, f will be an eigenvector of the operator A_{L_2} on $L_2(R)$. It follows in particular that A_{L_2} is *not* the operator of multiplication by the variable. In fact, in the contrary case f in (5.87) would be an eigenvector of the operator x on $L_2(R)$. We now have the following

Definition. If a proper basis e_H is such that the complete set of eigenvectors of the operator A in this basis belongs to H^* (alternatively, the complete set of eigenvectors of $\mathbf{A}$ belongs to $\mathbf{S}^*$), then the basis e_H is called the *string basis of eigenvectors of* $\mathbf{A}$ or the *string eigenbasis of* $\mathbf{A}$.

From the last example it must be clear that the string basis of eigenvectors does not always exist. In fact, the original operator x on $L_2(R)$ has no eigenvectors on this space. However, we also saw in the example that one can restrict the operator x to a Hilbert subspace H of $L_2(R)$ in such a way that the string eigenbasis e_H of this restriction exists. Simultaneously, the adjoint operator x^*, which is just the extension of x, is defined on the space $H^* \supset L_2(R)$. In [9] a general existence result to this effect is proven. In particular, this provides any Hermitian operator on a space L_2 with the self-adjoint extensions to larger Hilbert spaces of functions containing generalized eigenvectors of the operator.

For any self-adjoint operator A on the Euclidean space R^n there exists a basis on R^n consisting of the eigenvectors of A. We see that the coordinate formalism ensures a similar result on the string space $\mathbf{S}$ making the finite and the infinite-dimensional spaces even closer.

Let us now turn to the physical interpretation and illustration of the developed formalism.

6 The "Fourier transform" experiment

The developed formalism includes the standard QM formalism on a Hilbert space as a particular case. In fact, if we agree to stay within spaces of square-integrable functions only, the isometries become the unitary operators and the considered self-adjoint extensions of observables become the usual Hermitian operators. It is therefore expected that the formalism will be useful primarily in treating experiments with the improper states.

Let us begin with a mathematical description of a simple experiment with a magnetic spectrometer which involves evolution of improper states.

A free electron enters a constant magnetic field of magnitude B perpendicular to the electron's momentum vector p. Due to the Lorentz force the electron will move in a circle of radius $r = |p|/B$, where $|p|$ is the magnitude of p (here we neglect the effects related to the spin as well as the emission of photons due to acceleration of the electron).

By putting an absorbing scintillating screen on the electron path we will observe a single flash indicating the location of the electron at the moment of absorption. As electrons with the different momenta will move in circles of different radii we conclude that position of the electron at the moment of absorption is uniquely determined by p. Notice that in accordance with the uncertainty principle the momentum of the "landed" electron is completely uncertain.

Whenever the direction of motion of the electron changes to the opposite, the Lorentz force changes sign, showing that such an electron would be absorbed at the opposite side of the screen.

Long enough before the electron enters the field its wave function is the eigenfunction of the momentum operator, i.e. it is proportional to e^{ipx}, where x is the

coordinate along the electron path. At the moment of absorption the state function of the electron can be assumed to be an eigenfunction of the position operator, i.e. it is proportional to $\delta(y-p)$. Here y is the coordinate along the screen and the scale is chosen in such a way that the electron of momentum p is absorbed at the point with $y=p$.

On the other hand, the (inverse) Fourier transform F^{-1} of the generalized function e^{ipx} is:

$$F^{-1}[e^{ipx}](y)=\frac{1}{2\pi}\int e^{ipx}e^{-ixy}dx=\frac{1}{2\pi}\int e^{i(p-y)x}dx=\delta(y-p). \tag{6.88}$$

It follows that the state of the electron after it hits the screen is the Fourier transformation of the original state of the electron. Linearity of QM ensures then that any superposition of the free electrons will be "Fourier transformed" by the spectrometer into the corresponding superposition of the localized electrons.

As a result, the spectrometer transforms the space

$$H=\left\{\varphi:\varphi=\sum_{k=-\infty}^{\infty}c_k e^{ikx}, c_k\in R\right\} \tag{6.89}$$

into the space

$$\widetilde{H}=\left\{\psi:\psi=\sum_{k=-\infty}^{\infty}c_k\delta(x-k), c_k\in R\right\}. \tag{6.90}$$

Both the initial and the final states of the electron in the experiment are not square-integrable. However, the reason for that is different for these states. The position eigenstate $\delta(y-p)$ is not square integrable because of its "bad" behavior at $y=p$. The free electron state instead, is not square integrable because the state function e^{ipx} does not "fall off" for large $|x|$. The space $\widetilde{H}$ can be then made Hilbert if a "smoothened" version of the metric on the space $L_2(R)$ of square-integrable functions on R is used. In particular, in section 3 the space $\widetilde{H}$ furnished with the metric (3.21) was shown to be Hilbert. Similarly, the space H can be made Hilbert if a "falling off" Hilbert metric is used in place of the $L_2(R)$-metric.

In fact, the Fourier transform F establishes a one-to-one correspondence between the spaces H and $\widetilde{H}$. In particular, it induces a Hilbert structure on H by setting

$$(F\psi_1,F\psi_2)_H=(\psi_1,\psi_2)_{\widetilde{H}} \tag{6.91}$$

for any $\psi_1,\psi_2\in\widetilde{H}$. If H is furnished with the induced Hilbert structure, then F becomes an isomorphism of Hilbert spaces $\widetilde{H}$ and H. Namely, by (2.18), the action of $F:\widetilde{H}\longrightarrow H$ induces on H the following metric:

$$\begin{aligned}g_H(k,z)=\left(F^{-1*}\widehat{G}_{\widetilde{H}}F^{-1}\right)(k,z)&=\frac{1}{(2\pi)^2}\int e^{ikx}\sqrt{\frac{\pi}{2}}e^{-\frac{(x-y)^2}{2}}e^{-iyz}dxdy\\&=\frac{1}{2}\delta(k-z)e^{-\frac{k^2}{4}},\end{aligned} \tag{6.92}$$

where the metric $\widehat{G}_{\widetilde{H}}$ given by (3.22) has been used.

We see now that the entire process of evolution in the experiment can be viewed as a transformation of a state $\sum c_k e^{ikx}$ in the space H with the metric $g_H = \frac{1}{2}\delta(k-z)e^{-\frac{k^2}{4}}$ into the state $\sum c_k \delta(x-k)$ in the space $\widetilde{H}$ with the metric $\sqrt{\frac{\pi}{2}}e^{-(x-y)^2}$.

If desirable, one could even speculate on the physical significance of both metrics. The metric g_H contains the "cut off" factor $e^{-\frac{k^2}{4}}$ which indicates that the incoming electron cannot be considered free. One could associate the "deformation" of the L_2-metric $\delta(k-z)$ by the factor $e^{-\frac{k^2}{4}}$ with the influence of the spectrometer and the source of electrons.

The "smoothing" of the L_2-metric at the moment of absorption of the electron by the scintillating screen can be related to the fact that the final electron is only localized within some region (say, within an atom of the screen). The "distributed" metric $g_{\widetilde{H}}$ may account for the fact that this region has a positive Lebesgue measure.

The evolution operator in the experiment is given by the Fourier transform. Notice that the evolution operator is not rigorously speaking unitary. In fact, it is an isomorphism of *different* Hilbert spaces of functions H and $\widetilde{H}$. Nevertheless, the norm of a state is preserved under the evolution, i.e. the evolution operator is an *isometry*. The isometries represent the most obvious generalization of the unitary operators. Whenever an isometry preserves the Hilbert structure of its space of definition it becomes a unitary transformation. We now see in practice how an isometric evolution operator can be useful for a simple and mathematically rigorous treatment of evolution of improper states.

Notice that in the process of evolution the observable in the experiment changes: the magnetic field measures the electron's momentum while the scintillating screen measures its position. We see in particular that, not only the state function, but the equation for this function itself transforms in a covariant fashion in the experiment.

We conclude that the entire process can be described by a single functional tensor equation (4.59). In the considered case it can be in fact regarded as just a functional transformation which changes the component form of the eigenvalue problem (4.59).

7 The heavy quantum harmonic oscillator

In this section we are looking for a physically motivated example of a Hilbert space which contains functions even more singular than the delta-function. For this we set up a thought experiment in which position of a heavy particle will be measured.

To measure position of a massive particle consider a microscope where high energy photons are scattered off the particle. Assume for simplicity that the particle in the field of photons behaves as a quantum harmonic oscillator described by the Hamiltonian

$$\widehat{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}. \tag{7.93}$$

Here p and x are the momentum and position operators respectively, m is the mass and ω is the frequency of oscillations.

The n-th eigenstate ψ_n of such an oscillator is given by

$$\psi_n(x) = N_n e^{-\frac{m\omega x^2}{2\hbar}} H_n(x\sqrt{\frac{m\omega}{\hbar}}), \tag{7.94}$$

where

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n e^{-\xi^2}}{d\xi^n} \tag{7.95}$$

is the Hermite polynomial of order n, N_n is a normalization constant which depends on both m and ω and $\hbar$ is the Planck constant.

Therefore, to minimize the uncertainty of measuring position of the particle we need to increase the value of the product $m\omega$. Increasing ω corresponds to the increase in energy of the scattered photons. In the non-relativistic QM ω can in principle go to infinity producing a state concentrated at a point. In QFT this is not true. Due to generation of new particles the minimal uncertainty Δx of measuring coordinates of a particle in QFT is estimated by

$$\Delta x \sim \frac{\hbar}{mc}. \tag{7.96}$$

Suppose, however, that instead of increasing the frequency we take particles of increasingly large masses. Then, in principle, within the applicability of QFT we can localize the particle as well as we wish by taking m to be large enough.

Now, localized at a point in space states considered in QM are eigenfunctions of the position operator. We therefore expect that for any fixed n the n-th state of a heavy oscillator can be described by the δ-function.

To verify this conclusion mathematically we need to show that for any given n the sequence of states $\psi_n(x) = \psi_n(\alpha, x)$ parametrized for convenience by $\alpha = \sqrt{\frac{m\omega}{2\hbar}}$ is a delta-convergent sequence in the following sense (see also [1]):

Definition. A sequence of smooth (for simplicity) functions $f_\alpha(x)$ on the set of real numbers R is a delta-convergent sequence if for any fixed $a < 0, b > 0$

$$\lim_{\alpha \longrightarrow \infty} \int_a^b f_\alpha(x)dx = 1, \tag{7.97}$$

and for any fixed $a > 0, b > 0$ or $a < 0, b < 0$, the limit in (7.97) is zero. We will denote this fact by writing

$$f_\alpha(x) \longrightarrow \delta(x). \tag{7.98}$$

First of all, if the constants N_n in (7.94) are determined from the unit norm condition, the sequence $\psi_n(\alpha, x)$ cannot be a delta-convergent sequence. In fact, if $f_\alpha(x) \longrightarrow \delta(x)$ then the sequence of integrals $\int f_\alpha^2(x)dx$ is not bounded.

However, if the final state is to be given by the δ-function, the unit norm condition is not possible anyway. Let us therefore drop the unit norm condition and assume an arbitrary choice of the constants N_n.

It is known that the sequence

$$f_\alpha(x) = \frac{\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2} \tag{7.99}$$

is a delta-convergent sequence as $\alpha \longrightarrow \infty$. Therefore, assuming in (7.94) that $N_0 = \sqrt{\frac{m\omega}{2\pi\hbar}}$ and $\alpha = \sqrt{\frac{m\omega}{2\hbar}}$, we have for the ground state ψ_0:

$$\psi_0(x) = N_0 e^{-\alpha^2 x^2} \longrightarrow \delta(x) \tag{7.100}$$

as $\alpha \longrightarrow \infty$. That is, the ground state is a delta-convergent sequence of α .

However, the state ψ_1 is given by an odd function and thus cannot form a delta-convergent sequence. Therefore, not every state of the oscillator will yield the δ-state as $\alpha \longrightarrow \infty$.

Notice that the sequence obtained by the p-fold differentiation of the terms of a delta-convergent sequence converges to the p-th derivative $\delta^{(p)}(x)$ of the δ-function. Together with the fact that (7.99) is a δ-convergent sequence this means that

$$\frac{\alpha^{n+1}}{\sqrt{\pi}} \frac{d^p e^{-\alpha^2 x^2}}{d(\alpha x)^p} \longrightarrow \delta^{(p)}(x) \tag{7.101}$$

as $\alpha \longrightarrow \infty$. Formula (7.95) gives:

$$\frac{\alpha^{n+1}}{\sqrt{\pi}} \frac{d^p e^{-\alpha^2 x^2}}{d(\alpha x)^p} = \frac{(-1)^n \alpha^{n+1}}{\sqrt{\pi}} H_n(\alpha x) e^{-\alpha^2 x^2}. \tag{7.102}$$

On another hand, by (7.94) the n-th state of the oscillator is given by

$$\psi_n(x) = N_n e^{-\alpha^2 x^2} H_n(\sqrt{2}\alpha x). \tag{7.103}$$

From (7.101), (7.102), and (7.103) it follows that ψ_n is a linear combination of the delta-convergent sequence (7.99) and its derivatives. In particular, under an appropriate choice of the normalization constants N_n, as $\alpha \longrightarrow \infty$, ψ_n converges to $\delta^{(n)}(x)$ acting on an appropriate space of functions.

The generalized functions $\delta^{(n)}(x)$ are continuous functionals concentrated at the origin, i.e. equal to zero on a neighborhood of any point but the origin. Furthermore, any generalized function concentrated at a point is a finite linear combination of the δ-function and its derivatives [2].

The similarity of all functionals concentrated at a point makes one wonder why QM recognizes the δ-states but discriminates against other concentrated at a point states. One can of course deny the physical significance of states that correspond to α approaching infinity. In this case, however, we should dismiss the position

eigenstates as well. In fact, they are the "non-physical" limits of the physical states (wave packets) as the size of a packet approaches zero.

Mathematically, what we have here is similar to the process of completion needed to obtain the Hilbert space L_2 of square-integrable functions. This time we have a sequence of integrable functions which does not have a limit in L_2. One could say that convergence of such a sequence is too weak for the limit to be in L_2. By an appropriate generalization of the metric on the space L_2 we can include such limits into a Hilbert space. Indeed, in example 2 of section 3 it was verified that the space

$$H = \left\{ \varphi : \varphi = \sum_{k=0}^{\infty} c_k \delta^{(k)}(x), c_k \in R \right\} \tag{7.104}$$

with the metric (3.30) is Hilbert.

Let us remark that the obtained singular states are not necessary for the formalism developed in the paper. The formalism, however, allows for spaces containing various singular generalized functions. The considered experiment demonstrates that even the spaces containing such highly singular functions as the delta-function and its derivatives may be physically relevant.

In the next example a measurement of the electron's position in the two-slit experiment will be analyzed. We begin, however, with a brief review of a generally accepted mathematical approach to quantum measurement.

8 A mathematical model of quantum measurement

To see how to treat a quantum measurement in the string formalism let us begin with a simple description of the main mathematical model of quantum measurement. In this we will closely follow [6] (see also references therein).

A quantum measurement in continuous spectrum can be described with the use of a natural generalization of von Neumann projectors. Namely, a fuzzy (i.e. with a finite resolution) measurement with a continuous set of alternatives is represented mathematically by a multiplication of the state function $\psi(a)$ of the system in A-representation by a characteristic function $R_{a'}(a)$ of the measurement. The functions $R_{a'}$ satisfy the "decomposition of unity" condition

$$\int da' R^*_{a'} R_{a'} = 1. \tag{8.105}$$

Most typically, $R_{a'}(a) = R(a - a')$, where $R(a)$ is the Gaussian function, i.e.

$$R(a) = N e^{-\frac{a^2}{4\lambda^2}}, \tag{8.106}$$

where N is a normalization constant and λ is the resolution of the measurement.

The measurement is therefore represented by the scheme

$$\psi \longrightarrow \psi_{a'} = \widehat{R}_{a'} \psi, \tag{8.107}$$

where $\widehat{R}_{a'}$ is the operator of multiplication by the function $R_{a'}$ (which will be called the fuzzy projector). The probability density is given by

$$p_{a'} = \|\psi_{a'}\|^2 . \tag{8.108}$$

Notice that such a description can be used for discrete spectra as well. In fact, assume A is an observable with a spectrum a_i. Then

$$\psi(a) = \sum_i c_i \psi_i(a) \tag{8.109}$$

where ψ_i form an orthonormal basis of eigenfunctions of A. The measurement is then described by

$$\widehat{R}_{a'}\psi = \sum_i R(a_i - a')c_i\psi_i. \tag{8.110}$$

For a sharp enough measurement the characteristic function $R(a_i - a')$ can be taken to be 1 at a_i and 0 everywhere else in which case $\widehat{R}_{a'}$ becomes the von Neumann projector on ψ_i.

More realistically the model above with the Gaussian characteristic function describes a fuzzy measurement (i.e. the fuzzy projection) rather than a projection. However, the final result of a sequence of repeated instantaneous fuzzy measurements (8.110) is represented by the von Neumann projection.

Let us reproduce this result following [6]. Assume for simplicity that the observable A has two eigenvalues a_1, a_2 and ψ_1, ψ_2 are the corresponding eigenvectors. If the initial state of the system is given by

$$\psi = c_1\psi_1 + c_2\psi_2, \tag{8.111}$$

then after a single measurement (8.107) giving the result a' the state of the system is

$$\psi_{a'} = R(a_1 - a')c_1\psi_1 + R(a_2 - a')c_2\psi_2. \tag{8.112}$$

According to (8.108) the probability density of this result is given by

$$\|\psi_{a'}\|^2 = |\alpha_1|^2 |c_1|^2 + |\beta_1|^2 |c_2|^2 , \tag{8.113}$$

where $\alpha_1 = R(a_1 - a')$ and $\beta_1 = R(a_2 - a')$. The sequence of such measurements yields the sequence of states

$$c_1\psi_1 + c_2\psi_2 \longrightarrow \alpha_1 c_1\psi_1 + \beta_1 c_2\psi_2 \longrightarrow ... \longrightarrow \alpha_n ... \alpha_1 c_1\psi_1 + \beta_n ... \beta_1 c_2\psi_2 = \widetilde{c}_1\psi_1 + \widetilde{c}_2\psi_2. \tag{8.114}$$

It is important that only ψ_1 and ψ_2 are stable under the repeated measurements. In fact, assume for example that at some step of the process the coefficient of ψ_1 is larger than the coefficient of ψ_2. Then the probability is high that it will increase even more in the next step. As a result, one of the components dies out.

Finally, under the assumption that the Hamiltonian of interaction between the system and the measuring device dominates the free Hamiltonians of the system and the device, the above model is quite generally realizable [6]. In the next section it will be also shown that the model naturally fits in the presented string formalism.

9 The "Washing interference out" experiment

The fact that the evolution operator in the formalism can be an isometry rather than just a unitary transformation is very important in approaching the non-unitary collapse processes. In particular, the language of the formalism permits one in principle to treat the non-unitary collapse on equal footing with the unitary evolution, i.e. as an isometric process.

To understand the situation better let us analyze a measurement of the electron's position in the two-slit experiment. This experiment is especially useful as it tests the string formalism in a finite dimensional setting with no direct presence of improper states. The experiment is also called "washing interference out" or "which way" experiment and the modern interpretation of it can be found in [6].

The initial state of the electron in the experiment is taken to be a superposition of two localized states each corresponding to the electron going through each of the slits. Assume that

$$\psi(q) = c_1 N e^{-\frac{(q+a)^2}{4\varepsilon^2}} + c_2 N e^{-\frac{(q-a)^2}{4\varepsilon^2}} = c_1\psi_a(q) + c_2\psi_{-a}(q), \tag{9.115}$$

where $\pm a$ are coordinates of the slits, $\psi_{\pm a}(q) = N e^{-\frac{(q\pm a)^2}{4\epsilon^2}}$, N is a normalization constant (so that $\psi_{\pm a}(q)$ are unit-normalized) and $\varepsilon \ll a$.

Assume that a measurement or a sequence of fuzzy measurements is performed with the goal to find out which slit the particle went through. Each fuzzy measurement consists, for example, in a scattering of a photon off the electron.

Mathematically, as in section 8, each measurement will be described by the operator $\widehat{R}_x(q)$ of multiplication of ψ by the characteristic function $R_x(q)$ of the measurement. Although the exact form of $R_x(q)$ will not be important, we will assume that, as in (8.106), it is the Gaussian function i.e.

$$R_x(q) = M e^{-\frac{(q-x)^2}{4\lambda^2}}, \tag{9.116}$$

where M and λ are real numbers.

We then have after a single measurement

$$\widehat{R}_x(q)\psi(q) = M\left(c_1 e^{-\frac{(q-x)^2}{4\lambda^2}}\psi_a(q) + c_2 e^{-\frac{(q+x)^2}{4\lambda^2}}\psi_{-a}(q)\right). \tag{9.117}$$

Respectively, the probability density to obtain such a state is given by

$$p(x) = M^2\left(|c_1|^2 e^{-\frac{(q-x)^2}{2\lambda^2}} + |c_2|^2 e^{-\frac{(q+x)^2}{2\lambda^2}}\right). \tag{9.118}$$

The action of $\widehat{R}_x(q)$ in the basis (ψ_a, ψ_{-a}) is represented by the matrix

$$\rho = \begin{pmatrix} M e^{-\frac{(q-x)^2}{4\lambda^2}} & 0 \\ 0 & M e^{-\frac{(q+x)^2}{4\lambda^2}} \end{pmatrix}. \tag{9.119}$$

In the string formalism the operator $\widehat{R}_x(q)$ given by the matrix ρ is a functional transformation which changes not only the state ψ, but also the metric. Assume that the original metric (the "lab metric") in the experiment is represented by the 2×2 identity matrix. Then, by (2.18) at the moment of measurement the metric is

$$\widehat{G} = (\rho^* \rho)^{-1} = \begin{pmatrix} \frac{1}{M^2} e^{\frac{(q-x)^2}{2\lambda^2}} & 0 \\ 0 & \frac{1}{M^2} e^{\frac{(q+x)^2}{2\lambda^2}} \end{pmatrix}. \tag{9.120}$$

Assume now that the measurement is precise enough to resolve between the slits, i.e. $\lambda \ll a$. Then the probability density in (9.118) is not negligible only for $x = \pm a$. This means that only one of the two diagonal entries in (9.119) survives. The second, although different than zero, is much smaller.

Assuming for example that $x = a$, the transformation matrix ρ in (9.119) can be written as

$$\rho = M \begin{pmatrix} 1 - \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix}, \tag{9.121}$$

where $\epsilon_{1,2} \ll 1$. Transformation of $\psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ gives

$$\rho\psi = M \begin{pmatrix} 1 - \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = M \begin{pmatrix} (1 - \epsilon_1) c_1 \\ \epsilon_2 c_2 \end{pmatrix}. \tag{9.122}$$

As $\epsilon \ll 1$ we see that only ψ_a component is not negligible. Notice also that, as discussed in section 8, under the repeated measurements the second component becomes smaller and smaller approaching zero. The resulting transformation then looks like a von Neumann projector and the resulting state $\widetilde{\psi}$ looks like the eigenstate of the position operator.

On the other hand, it is very important that the determinant of ρ is never zero, that is, ρ can be considered as an isometry. That is, the norm of ψ as well as the inner product of any two vectors are preserved.

The presented analysis of the two-slit experiment can be now summarized as follows. In trying to determine which slit the electron went through we scatter the photons off the electron. Mathematically each scattering event can be described as a multiplication of the state function of the electron by the Gaussian function.

In the conventional approach to QT each scattering event is therefore a fuzzy projector. The sequence of scattering events leads to the gradual decoherence. The final result is represented by the von Neumann projection.

In the string formalism each scattering event is a (reversible) functional transformation which changes the state and the metric. In the new metric the norm of the new state is the same as the norm of the original state in the original metric. That is, the transformation is an isometry.

However, when the original metric (the lab metric) is used to describe the state of the electron, only one of the components of the state function survives. By (9.122) the second component is negligibly small in the lab metric.

This demonstrates the possibility to "collapse" various state functions in the experiment by applying an isometry that suppresses one of the components of the functions. The suppressed component can be made arbitrarily small for any given state by choosing an appropriate resolution λ in (9.119) or by applying the isometry ρ with a given resolution several times. Moreover, the suppressed component can vanish completely for all states at once in the "correct" infinite-dimensional approach to the problem. For, we could take the initial state function ψ to be an element of a more general space H of linear combinations of functions supported on a small enough neighborhood of the slits $x = \pm a$. The state after measuring the electron's position would be then an element of the Hilbert space $\widetilde{H}$ of functions localized on a small neighborhood of one of the slits $x = a$ or $x = -a$. In this case the measurement itself would be represented by an isomorphism

$$M : H \longrightarrow \widetilde{H}. \tag{9.123}$$

In particular, the final state would be exactly zero at one of the slits.

We point out that the above consideration does not explain the collapse, nor it provides us with any details of collapse as the dynamics of collapse is not present. However, the analysis suggests that the developed formalism is an appropriate *language* for treating the unitary and non-unitary processes in QM in a uniform way.

10 Summarizing the results

Let us briefly review the findings and make final comments.

First of all, the formalism has been developed in which to do quantum mechanics and which includes the improper state functions on equal footing with the square-integrable state functions. The formalism embraces the existing formalism of QM and in particular case of a fixed Hilbert model of square-integrable functions is capable of producing the same results.

The strength of the formalism is in its covariant structure. This structure permits to describe physics equally well in arbitrary functional coordinates including the coordinates represented by singular generalized functions. Already in the standard QT the states and the observables change according to (2.16), (2.19) under a given unitary representation of a symmetry group. Once it is accepted that the metric on the space to which a state function belongs is also transformed by (2.18), it becomes possible to include in the formalism arbitrary isometric transformations while preserving the tensor form of the equations. In particular, the choice of a Hilbert space in which to describe a quantum system becomes a matter of convenience only.

The formalism unifies different pictures of the quantum world. We saw in particular that the eigenvalue problems for the position and the momentum operators

are but two coordinate forms of a single functional tensor eigenvalue problem on the string space. One could say that the process of measuring position is essentially the same as the process of measuring momentum, but "observed" from a different system of functional coordinates. We also see that a state which is concentrated at a point in one coordinate system, can be represented by a smooth function in another one.

The observables in the formalism are in general extensions of the standard QM-observables. These extensions are self-adjoint (as defined in section 5) operators acting on an appropriate Hilbert spaces of functions $H^* \supset L_2$. It becomes then possible to include the generalized eigenvectors of such extensions into the space H^* itself.

Respectively, the unitary operators in the formalism are generalized to isometries. In particular, the formalism can describe the evolution of improper states equally well as the evolution of square-integrable states. It is therefore equally well adapted for description of the unitary evolution and non-unitary collapse processes.

The covariant structure makes the formalism different from the rigged Hilbert space approach to QM not only in mathematics involved, but also in spirit. The Gelfand's triples of spaces used in QM are constructed around an L_2-space (the middle term of a triple). The construction serves the purpose of a rigorous mathematical treatment of improper states in QM. However, the symmetry between the improper and the square-integrable states is lost. In particular, the norm of improper states is not defined . The necessity to carry three different spaces of functions to treat the improper states is awkward. More importantly, no general covariant description of the construction seems to be possible.

The main task of the paper was to develop the functional coordinate formalism and illustrate its usefulness in QT. Many questions remain open. In particular, the dynamics of improper states was not discussed. As a result, a particular choice of the Hilbert metric in physical experiments is not clear. More generally, it remains to be seen if the formalism is appropriate to serve other needs of quantum theory.

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REFERENCES

1. I.M. Gel'fand and G.E. Shilov, *Generalized Functions, Vol.1*, (Academic Press, New York and London, 1964).

2. I.M. Gel'fand and G.E. Shilov, *Generalized Functions, Vol.2*, (Academic Press, New York and London, 1968).
3. I.M. Gel'fand and N.Y. Vilenkin, *Generalized Functions, Vol.4*, (Academic Press, New York and London, 1964).
4. A. Bohm, *The Rigged Hilbert Space and Quantum Mechanics*, Lecture Notes in Physics, Vol.78, (Springer, New York, 1978).
5. S. Lang, *Differentiable manifolds*, (Springer-Verlag, New York, 1985).
6. M.B. Mensky, *Quantum Measurements and Decoherence*, (Kluwer Academic Publishers, Dordrecht, Boston, and London, 2000).
7. A. Kryukov, "Coordinate formalism on Hilbert manifolds", (arXiv: math-ph/0201017 v1, 2002).
8. A. Kryukov, "Coordinate formalism on abstract Hilbert space", (arXiv: math-ph/0201031 v1, 2002).
9. A. Kryukov, *Int. J. Math. & Math. Sci.* **14**, 2241 (2005)